Hyperbolic Geometry in Terms of Point-Reflections or of Line-Orthogonality

Victor Pambuccian

Department of Integrative Studies, Arizona State University West
Phoenix, AZ 85069-7100, U.S.A. e-mail: pamb@math.west.asu.edu

Abstract

We provide $\forall\exists$-axiom systems for n-dimensional hyperbolic geometry over Euclidean fields in terms of (i) points and the binary operation of point-reflection, and, for $n \neq 3$, in terms of (ii) lines and binary relation of line-orthogonality.

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1 Introduction

Hyperbolic geometry has so far been axiomatized in various ways. Hilbert [3], [4] axiomatized it in the same language he used to axiomatize Euclidean geometry, and so did Tarski and Szmielew [24]. Menger [11] showed that it can be axiomatized in terms of points and the ternary relation of collinearity, and an axiom system for the two-dimensional case has been provided in [22]. In [18] it was shown that plane hyperbolic geometry can be axiomatized by means of universal axioms in a language containing only individual constants and ternary operations, all axioms containing at most 4 variables. It can also be axiomatized in terms of lines and the binary relation of line-perpendicularity for all finite dimensions $n \neq 3$ (cf. [21], [19], [9], [10], [16]), and in the 3-dimensional case planes and plane-perpendicularity can serve as primitive notions (cf. [12]). Spheres and the binary relation of sphere-tangency can also serve as single primitive notion for all finite dimensions ≥ 2 (as shown in [17]; cf. [13] for a survey of other axiomatizations).

Given that in hyperbolic geometry of any dimension three points a, b, c are collinear if and only if the composition of the reflections in them, $\sigma_a\sigma_b\sigma_c$ (where by σ_x we have denoted the reflection in x), is a point-reflection (cf. [5], [6]), one obtains as an immediate corollary of the axiomatizability of hyperbolic geometry in terms of the collinearity relation that hyperbolic geometry of any dimension (even the dimension-free version, cf. [24, II.4.62]), is axiomatizable in terms of the binary operation of point-reflection. This is in stark contrast to Euclidean geometry, which, as shown in [2], cannot be axiomatized by means of any finite set of binary operations. However, if one were to obtain an axiom system for hyperbolic geometry in terms of point-reflections starting with an axiom system based on collinearity alone, and replacing every occurrence of the collinearity predicate with its definition in terms of σ, one would obtain an axiom system of quantifier complexity at least as high as that of the original axiom system. It was shown in [24] that there is no $\forall\exists$-axiom system for hyperbolic geometry in terms of collinearity, and in [**?**] that there is a $\forall\exists\forall\exists$ axiom system for it, but no $\forall\exists\forall$ one.

A plane hyperbolic geometry whose models are Kleinian inner-disc models in affine planes over *arbitrary ordered fields* (to be referred to in the sequel as KLINGENBERG*'s generalized hyperbolic planes*), the notions of betweenness and equidistance being interpreted just like in the standard hyperbolic case — betweenness coincides with affine betweenness, and equidistance is given by $\mathbf{ab} \equiv \mathbf{cd}$ if and only if $\Psi(\mathbf{a}, \mathbf{b}) = \Psi(\mathbf{c}, \mathbf{d})$, where $\Psi(\mathbf{x}, \mathbf{y}) = (1 - \mathbf{x} \cdot \mathbf{x})(1 - \mathbf{y} \cdot \mathbf{y})(1 - \mathbf{x} \cdot \mathbf{y})^{-2}$, and $\mathbf{u} \cdot \mathbf{v}$ is defined as $u_1v_1 + u_2v_2$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, all points $\mathbf{x} = (x_1, x_2)$ satisfying $x_1^2 + x_2^2 < 1$ — has been axiomatized by KLINGENBERG [7]. KLINGENBERG's axiomatics has been later simplified by BERGAU (cf. [1]).

The purpose of this note is: (i) to provide a $\forall\exists$-axiom system for both KLINGENBERG's generalized hyperbolic planes and for hyperbolic geometry over Euclidean ordered fields of arbitrary finite dimension based on point-reflections alone, and (ii) to provide a $\forall\exists$-axiom system in terms of lines and line-orthogonality for both KLINGENBERG's generalized hyperbolic planes and for hyperbolic planes over Euclidean ordered fields. Thus, in every axiom, when written in prenex form, all universal quantifiers (if any) precede all existential quantifiers (if any).

We shall obtain these axiom systems for plane hyperbolic geometry by using the axiom system provided by KLINGENBERG [7], with some changes due to BERGAU (cf. [1]). For (i), we shall use as axiom system for non-elliptic metric planes (i. e. metric planes in which the perpendicular from a point outside of a line to that line is unique (cf. [1, §3,4, Satz 5])) the one provided in [15].

Given that no finite set of binary relations with individual variables to be interpreted as *points* can axiomatize hyperbolic geometry (cf. [20], [24]), (i) is, from the point of view of the minimal arity of the notions involved, the simplest possible axiom system for hyperbolic geometry.

The significance of point reflections in both absolute and hyperbolic geometry has been studied in [5] and [6].

With (ii) we solve the open problem stated in [16].

2 The reflection based axiom system

The language $\mathcal{L}$ in which the axiom system will be expressed has only one sort of individual variables, to be interpreted as *points*, and a binary operation symbol σ, with $\sigma(ab)$ to be interpreted as *the reflection of b in a*. We shall also write $\sigma_a(b)$ for $\sigma(ab)$, and for improved readability and to reduce the number of parentheses, we shall write $\sigma_{x_1}\sigma_{x_2}\ldots\sigma_{x_n}(a)$ instead of $\sigma(x_1(\sigma(x_2(\ldots\sigma(x_na))\ldots)$.

In order to formulate the axioms in a more readable way, we shall use the following abbreviations:

$$\begin{aligned} L(x_1x_2x_3) \quad &:\Leftrightarrow \quad \bigwedge_{i=1}^{2} \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_1}\sigma_{x_2}\sigma_{x_3}(x_i) = x_i, \\ cd \perp_e ab \quad &:\Leftrightarrow \quad L(abd) \wedge L(cde) \wedge \sigma(e\sigma(bc)) = \sigma(\sigma(db)c), \end{aligned}$$

$$R_{mn}(uvaa') \quad :\Leftrightarrow \quad (L(uva) \wedge a' = a) \vee (\neg L(auv) \wedge an \perp_m uv \wedge a' = \sigma(na)).$$

Here $L(abc)$ stands for *the points* a, b, c *are collinear* (but not necessarily distinct), $cd \perp_e ab$ stands for d *is the footpoint of the perpendicular from* c *to* ab (e being an auxiliary point used in the construction), to be used only for c not collinear with a and b, and $R_{mn}(uvaa')$ stands for a *and* a' *are symmetric with respect to the line determined by* u *and* v (m and n designating auxiliary points used in the construction), to be used only for $u \neq v$.

For the definition of L, we have used the fact that an involutory product of three point symmetries must be a point symmetry ([1, §3,7, Satz 16, 17]), and the fact that a motion which fixes two different points x_1 and x_2 will have to fix all the points of the line determined by them, so it has to be either a reflection in that line or the identity. Since a product of six point symmetries is equal to a product of twelve, i. e. an even number of line symmetries, it cannot be a line symmetry, so it has to be the identity if it fixes two different points.

For the definition of the perpendicularity relation $\perp$ we have used the following

Proposition 1. *In any Hilbert plane (i. e. in every model of plane absolute geometry, which is a geometry satisfying the axioms A1–A9 from [24], or the axioms I 1–3, II, III from [3]) with non-Euclidean metric (i. e. in which there is no rectangle) we have: If the median* ce *to the side* $\sigma(bc)\sigma(\sigma(db)c)$ *of a triangle* $c\sigma(bc)\sigma(\sigma(db)c)$ *passes through the midpoint* d *of the line joining the midpoints* b *and* $\sigma(db)$ *of the other two sides, then the median* ce *is the altitude of the triangle, and thus perpendicular to the midline* $b\sigma(db)$ *as well.*

Proof. (due to JAROSŁAW KOSIOREK, see Fig. 1) Given that the metric is non-Euclidean the point $\sigma(dc)$ is different from e. The triangles $b\sigma(bc)\sigma(dc)$ and $\sigma(db)\sigma(dc)\sigma(\sigma(db)c)$ are congruent by side-angle-side, as $b\sigma(bc) \equiv bc \equiv \sigma(db)\sigma(dc)$, $b\sigma(dc) \equiv \sigma(db)c \equiv \sigma(db)\sigma(\sigma(db)c)$ and the angles at b and $\sigma(db)$ are congruent given that their supplements are the sum of two congruent angles. Given that the two triangles are congruent, we have $\sigma(bc)\sigma(dc) \equiv \sigma(dc)\sigma(\sigma(db)c)$, thus $\sigma(dc)e$ is the perpendicular bisector of $\sigma(bc)\sigma(\sigma(db)c)$. Thus $c\sigma(bc) \equiv c\sigma(\sigma(db)c)$, so $cb \equiv c\sigma(db)$ as well, and cd is perpendicular to $b\sigma(db)$. □

We are now ready to state the axioms, in which we will omit universal quantifiers for universal axioms.

A 1 $\sigma(aa) = a$

A 2 $\sigma(a\sigma(ab)) = b$

A 3 $\sigma(ax) = \sigma(bx) \rightarrow a = b$

A 4 $L(abc) \rightarrow L(cba) \wedge L(bac)$

A 5 $a \neq b \wedge L(abc) \wedge L(abd) \rightarrow L(acd)$

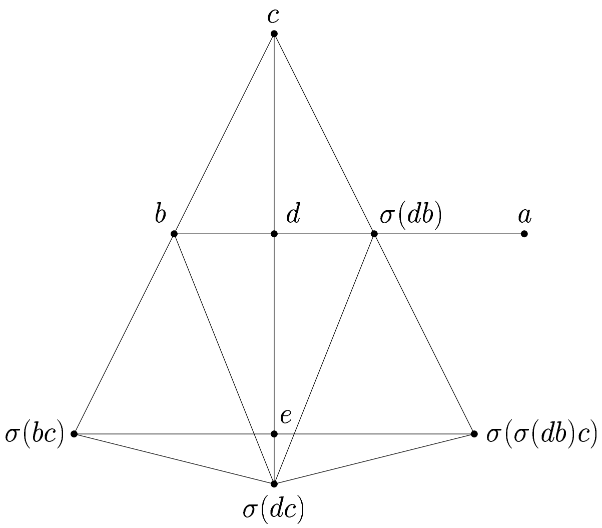

Figure 1: The definition of perpendicularity in terms of σ

A 6 $L(ab\sigma(ab))$

A 7 $\sigma(\sigma(xa)\sigma(xb)) = \sigma(x\sigma(ab))$

A 8 $(\forall abc)(\exists de)\,\neg L(abc) \to cd \perp_e ab$

A 9 $\neg L(abx) \wedge xd \perp_e ab \wedge \neg L(aby) \wedge yd \perp_f ab \to L(xyd)$

A 10 $\neg L(abx) \wedge c \neq d \wedge L(abc) \wedge L(abd) \wedge xu \perp_e ab \wedge xv \perp_f cd \to u = v$

A 11 $\neg L(abx) \wedge xu \perp_e ab \wedge L(xyu) \wedge y \neq u \wedge yv \perp_f ab \to u = v$

A 12 $\neg L(abc) \wedge ca \perp_e ab \wedge bu \perp_f ac \to u = a$

A 13 $\neg L(axy) \wedge yx \perp_e xa \wedge \sigma(ay)u \perp_e xa \to u = \sigma(ax)$

A 14 $\neg L(abc) \wedge u \neq v \wedge R_{mn}(uvaa') \wedge R_{pq}(uvbb') \wedge R_{rs}(uvcc') \to \neg L(a'b'c')$

A 15 $\neg L(abc) \wedge u \neq v \wedge ca \perp_e ab \wedge R_{mn}(uvaa') \wedge R_{pq}(uvbb') \wedge R_{rs}(uvcc')$
$\wedge c'x \perp_f a'b' \to x = a'$

A 16 $\neg L(oab) \wedge x \neq o \wedge \neg L(obc) \wedge R_{mn}(oaxy) \wedge R_{pq}(obyz) \wedge R_{rs}(oczt)$
$\wedge((\neg L(xto) \wedge oo' \perp_e xt) \vee (o = o' \wedge L(xto) \wedge x \neq t)) \to \sigma(o'x) = t$

A 17 $\neg L(oab) \wedge \neg L(obc) \wedge \bigwedge_{i=1}^{2}(R_{m_i n_i}(oax_i y_i) \wedge R_{p_i q_i}(oby_i z_i)$
$\wedge R_{r_i s_i}(ocz_i t_i) \wedge \sigma(g_i x_i) = t_i) \to L(og_1 g_2)$

A 18 (i) $L(abc) \wedge a \neq b \wedge b \neq c \wedge \neg L(aba') \wedge a'a \perp_m ab \wedge \neg L(abb') \wedge b'b \perp_n ab$
$\wedge \neg L(abc') \wedge c'c \perp_p ab \wedge \neg L(abx) \wedge R_{ef}(aa'xy) \wedge R_{gh}(bb'yz) \wedge R_{kl}(cc'zt)$
$\wedge L(abu) \rightarrow \neg L(xtu)$

(ii) $L(abc) \wedge a \neq b \wedge b \neq c \wedge \neg L(aba') \wedge a'a \perp_m ab \wedge \neg L(abb') \wedge b'b \perp_n ab$
$\wedge \neg L(abc') \wedge c'c \perp_p ab \wedge \sigma(oa) = \sigma(c\sigma(ba)) \wedge \neg L(abx) \wedge R_{ef}(aa'xy) \wedge R_{gh}(bb'yz)$
$\wedge ou \perp_q xt \wedge uw \perp_r ab \rightarrow t = \sigma(ux) \wedge w = o$

A 19 (i) $(\forall abc)(\exists o)\, L(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a$
$\rightarrow \sigma(oa) = \sigma(c\sigma(ba))$

(ii) $a \neq b \wedge b \neq c \wedge c \neq a \wedge L(abc) \wedge L(abx)$
$\wedge \sigma(oa) = \sigma(c\sigma(ba)) \rightarrow \sigma(c\sigma(b\sigma(ax))) = \sigma(ox)$

A 20 $(\forall oabp_1p_2p_3)(\exists xymn)\, \neg L(oab) \wedge \bigwedge_{i=1}^{3} \neg(o = p_i) \rightarrow \bigvee_{i=1}^{3}((L(op_ix) \wedge L(abx))$
$\vee(\neg L(op_iy) \wedge yx \perp_m op_i \wedge \neg L(abx) \wedge xy \perp_n ab) \vee L(op_ip_{i+1}))$

A 21 $(\exists abcdeouvwxy)\, \neg L(abd) \wedge ab \perp_u bd \wedge \neg L(oab) \wedge oa \perp_v ab \wedge \neg L(oca) \wedge oc \perp_w ca$
$\wedge \neg L(odb) \wedge od \perp_x db \wedge ce \perp_y oa \wedge L(aeo) \wedge L(ced)$

A 22 $(\forall ab)(\exists m)\, \sigma(ma) = b$

The axioms make the following statements: A1: the point a is a fixed point of the reflection σ_a, A2: reflections in points are involutory transformations (or the identity); A3: reflections of a point in two different points do not coincide; A4: collinearity of three points is a symmetric relation; A5: if c and d are collinear with a and b, then a, c, d are collinear as well; A6: the reflection of b in a is collinear with a and b; A7: reflections in points preserve midpoints; A8: from every point outside of a line there is a perpendicular to that line; A9: if the footpoints of two perpendiculars to a line coincide, then the perpendiculars themselves coincide; A10: the footpoint of the perpendicular from x to the line ab does not depend on the particular choice of points a and b that determine the line ab; A11: if x is a point outside of the line ab, and y is a point on the perpendicular from x to ab, then the footpoints of the perpendiculars of x and y to the line ab coincide; A12 states that perpendicularity is a symmetric relation (if ca is perpendicular to ab, then ba is perpendicular to ac); A13: if yx is perpendicular to xa, the so are $\sigma_a(y)\sigma_a(x)$ and $\sigma_a(x)a$; A14 and A15 state that reflections in lines preserve the non-collinearity and the orthogonality relation; A16 and A17: the composition of the reflections in three lines oa, ob, oc that have the point o in common is a reflection in a line through o; A18 and A19 state that the composition of three reflections in the lines aa', bb', cc' perpendicular to the line ab is a reflection in a line perpendicular to ab as well; A20: from a point o not on line ab there are at most two different lines that have neither a point nor a perpendicular in common with ab; A21: There are two lines, ac and bd, which are hyperbolically parallel according to BERGAU's criterion (cf. [1, §14,3, p. 224, Kriterium], see Fig. 2); A22: every segment has a midpoint.

With $\Sigma = \{\text{A1-A21}\}$ and $\Sigma' =, \{\text{A1-A22}\}$, we have the following

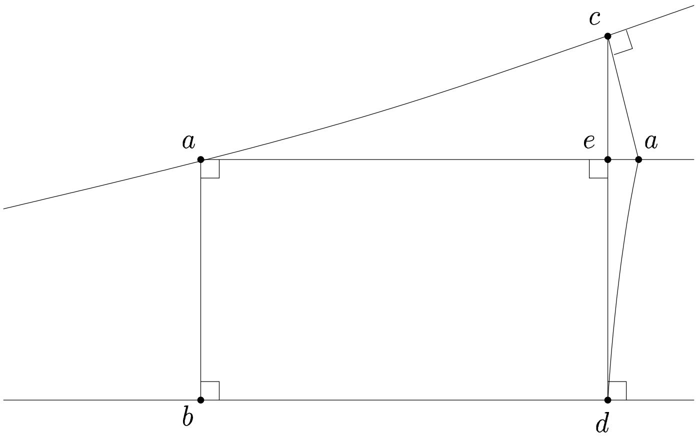

Figure 2: Bergau's criterion for parallelism

Theorem 1. *Σ and Σ' are axiom systems for Klingenberg's generalized hyperbolic planes and for hyperbolic planes over Euclidean fields. In every model of Σ or Σ' the operation σ has the intended interpretation.*

Proof. The axioms A1-A18, as well as the existence of three non-collinear points, which is part of the statement of A21, imply — with the obvious definitions of the footpoint operation F and of the fourth reflection point operation π ($\pi(abc)$ will be $\mu(a\sigma(c\sigma(ba)))$), where $\mu(xy)$ stands for the midpoint of the segment xy) which are part of the language in which the axiom system from [15] is expressed — the axioms given therein for non-elliptic metric planes. Thus σ has the desired interpretation. By Bergau's lemma [1, §14,3], the two lines ac and bd from A21 have neither a point nor a perpendicular in common, so the non-elliptic metric plane that satisfies the axioms of Σ satisfies the two axioms V* and H of [1, §14,1], making it a generalized hyperbolic plane in the sense of Klingenberg [7]. Thus Σ is an axiom system for them. In the case of Σ', since every segment has, by A22, a midpoint, the coordinate field of the generalized hyperbolic plane must be a Euclidean field, as proved in [7, p. 352], and thus is the regular hyperbolic plane, with Kleinian inner-circle models over Euclidean ordered fields as models. □

3 The orthogonality based axiom system

We now turn to the axioms of a $\forall\exists$-axiom system for plane hyperbolic geometry with individual variables to be interpreted as *lines* and with the binary relation of *line-orthogonality* as the only primitive notion.

To both express the axioms in a more readable manner and to significantly shorten them, we shall introduce a number of abbreviations. We write $\neq (a_1 \dots a_n)$ for $\bigwedge_{1\leq i<j\leq n} a_i \neq a_j$, as well as $(a \perp b_1, \dots, b_n)$ for $\bigwedge_{i=1}^{n} a \perp b_i$, and we use the following abbreviations:

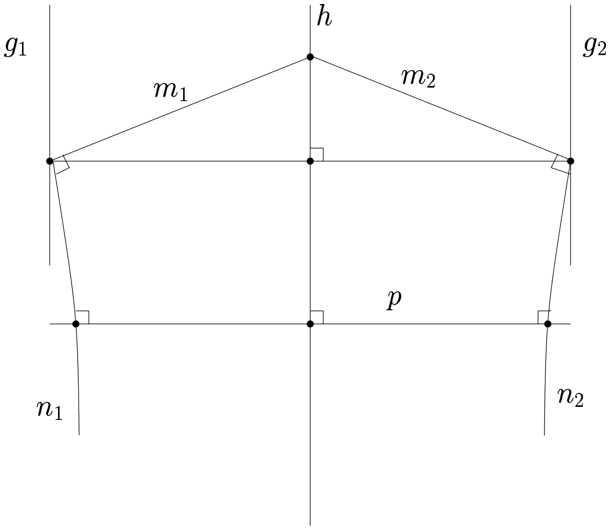

Figure 3: The definition of ϱ

$$
\begin{aligned}
\alpha(abc) \quad &:\Leftrightarrow \quad (\exists h) \neq (abc) \wedge (h \perp a, b, c) \\
\pi(a_1a_2a_3) \quad &:\Leftrightarrow \quad (\exists op_1p_2p_3q_1q_2q_3) \bigwedge_{i=1}^{3} (\alpha(a_ip_ip_{i+1}) \wedge \alpha(a_iq_iq_{i+1}) \wedge \alpha(op_iq_i)) \\
\overline{\pi}(abc) \quad &:\Leftrightarrow \quad \pi(abc) \vee c = a \vee c = b \\
\varrho(g_1hg_2lm_1m_2n_1n_2p) \quad &:\Leftrightarrow \quad h \perp l \wedge \pi(m_1m_2h) \wedge \bigwedge_{i=1}^{2} (g_i \perp l \wedge \pi(g_im_il) \wedge \pi(g_im_in_i) \wedge (n_i \perp m_i, p)) \\
\zeta(g_1hg_2l) \quad &:\Leftrightarrow \quad (\exists m_1m_2n_1n_2p)\, \varrho(g_1hg_2lm_1m_2n_1n_2p) \\
\overline{\zeta}(g_1hg_2l) \quad &:\Leftrightarrow \quad \zeta(g_1hg_2l) \vee (g_1 = h \wedge g_2 = h \wedge l \perp h) \\
\gamma(abctlt'w'e) \quad &:\Leftrightarrow \quad (\exists uu'vv'wmn)\, \overline{\zeta}(taul) \wedge (\pi(ulm) \vee m = u) \wedge \pi(luu') \wedge \overline{\zeta}(u'bvm) \\
&\qquad \wedge \pi(mvv') \wedge (\pi(vmn) \vee n = v) \wedge \overline{\zeta}(v'cwn) \wedge \overline{\pi}(tlt') \wedge \overline{\pi}(wnw') \\
&\qquad \wedge \overline{\pi}(tle) \wedge \overline{\pi}(wne) \wedge (e \perp t', w') \wedge \neg(\overline{\pi}(tlw) \wedge \overline{\pi}(tlw)) \\
\varphi(abctl) \quad &:\Leftrightarrow \quad (\exists uu'vv'wmn)\, \overline{\zeta}(taul) \wedge (\pi(ulm) \vee m = u) \wedge \pi(luu') \\
&\qquad \wedge \overline{\zeta}(u'bvm) \wedge \pi(mvv') \wedge (\pi(vmn) \vee n = v) \wedge \overline{\zeta}(v'cwn) \wedge \overline{\pi}(tlw) \wedge \overline{\pi}(tlw)
\end{aligned}
$$

Here $\alpha(abc)$ stands for a, b, c *are three different lines which have a common perpendicular* (or equivalently which have the same pole), $\pi(abc)$ stands for a, b, c *are three different lines which meet in a point* (see [19] or [21] for a proof that this is indeed the case), $\overline{\pi}(abc)$ for a, b, c *are three concurrent lines or* $c \in \{a, b\}$.

We shall think of pairs of orthogonal lines (a, b) as a *point*, and we shall say that the line l passes through the point (a, b) if and only if $\overline{\pi}(abl)$, and that points (a, b) and (c, d) are *equal* if and only if $\overline{\pi}(abc) \wedge \overline{\pi}(abd)$. In particular the point (a, b) is always equal to the point (b, a).

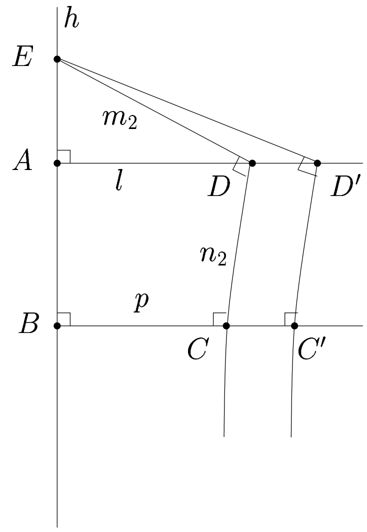

Figure 4: The sum of the angles of $DCC'D'$ would be greater than 360°

Of the remaining defined notions, $\varrho(g_1hg_2lm_1m_2n_1n_2p)$ and $\zeta(g_1hg_2l)$ stand for *g_1, g_2, and h are three different lines which are perpendicular to l, and g_2 is the reflection of g_1 in h*, $\overline{\zeta}(g_1hg_2l)$ stands for *g_2 is the reflection of g_1 in h and l is their common perpendicular* — an we shall denote for the purpose of explaining the above abbreviations the reflection in a line h by σ_h, $\gamma(abctlt'w'e)$ stands for *e is the line joining the point (l,t) with its image (w',e) under $\sigma_c\sigma_b\sigma_a$, (w',e) being a point different from (l,t), and l being perpendicular to a*, to be used only when a,b,c have a point or a perpendicular in common, and in the latter case l must be different from the common perpendicular, and finally $\varphi(abctl)$ stands for *(t,l) is a fixed point of $\sigma_c\sigma_b\sigma_a$*, to be used only in the cases mentioned above.

To see that the definition of ϱ does imply that the point of intersection of h and l is the midpoint of the segment formed by the points of intersection of g_1 and g_2 with l, let A, B, C, D, M, N, and E denote the points of intersection of h and l, h and p, p and n_2, l and n_2, l and n_1, p and n_1, and h and m_2 respectively. Suppose that AD is not congruent to AM. W. l. o. g. we may assume that AM is greater than AD. Let D' and C' be the reflections of M and N in h. Since $\angle ADE > \angle AD'E$ and both $\angle EDC$ and $\angle ED'C'$ are right, we must have $\angle ADC < \angle AD'C'$, which implies that the sum of the angles of the convex quadrilateral $DCC'D'$ exceeds 360°, a contradiction.

Given g_1, h, g_2, l, such that g_2 is the reflection of g_1 in h, it is easy to show that one can indeed find m_i, n_i for $i = 1, 2$ and p such that $\varrho(g_1hg_2lm_1m_2n_1n_2p)$ holds. If A, D, M denote the same intersection points as above, then by choosing the point E close enough to A we can always ensure that the perpendiculars n_2 and n_1 raised in D and M to ED and EM respectively do not intersect h, and thus h and n_1 must have a common perpendicular p, which must be perpendicular to n_2 as well, given that n_2 is the reflection of n_1 in h.

Consider the following axioms:

O 1 $g \perp h \rightarrow h \perp g$

O 2 $(\bigwedge_{i=1}^{2} \bigwedge_{j=1}^{2} l_i \perp g_j) \to (l_1 = l_2 \vee g_1 = g_2)$

O 3 $(\forall g_1 h l)(\exists g_2 m_1 m_2 n_1 n_2 p)\, (l \perp g_1, h) \wedge g_1 \neq h \to g_2 \perp l \wedge \pi(m_1 m_2 h)$
$\wedge \bigwedge_{i=1}^{2}(\pi(g_i m_i l) \wedge \pi(g_i m_i n_i) \wedge (m_i, p \perp n_i))$

O 4 $\zeta(g_1 h g_2 l) \wedge \zeta(g_1 h g_2' l) \to g_2 = g_2'$

O 5 $(\forall abce)(\exists d)(a \perp b \vee \pi(abc)) \to d \perp e \wedge \overline{\pi}(abd)$

O 6 $(m \perp n \vee \pi(mno)) \wedge \overline{\pi}(mnp) \wedge \overline{\pi}(mnp') \wedge (p, p' \perp l) \to p = p'$

O 7 $(\forall abcd)(\exists m)\, a \perp b \wedge c \perp d \wedge \neg\overline{\pi}(abc) \to \overline{\pi}(abm) \wedge \overline{\pi}(cdm)$

O 8 $a \perp b \wedge c \perp d \wedge \neg\overline{\pi}(abc) \wedge \bigwedge_{i=1}^{2}(\overline{\pi}(abm_i) \wedge \overline{\pi}(cdm_i)) \to m_1 = m_2$

O 9 $(\forall g_1 g_2 g_3 g_1' g_2' g_3' l_1 l_2 l_3 h m)(\exists m')\ \bigwedge_{i=1}^{3}(\overline{\zeta}(g_i h g_i' l_i) \wedge \overline{\pi}(g_i l_i m) \wedge \neg\overline{\pi}(g_i l_i g_{i+1})) \to \bigwedge_{i=1}^{3} \overline{\pi}(g_i' l_i m')$

O 10 $(\bigwedge_{i=1}^{3} \overline{\zeta}(g_i h g_i' l_i) \wedge \neg\overline{\pi}(g_i l_i g_{i+1})) \wedge \overline{\pi}(g_1 l_1 m) \wedge \overline{\pi}(g_2 l_2 m) \wedge \overline{\pi}(g_1 l_1 n) \wedge \overline{\pi}(g_3 l_3 n)$
$\wedge m \perp n \wedge \overline{\pi}(g_1' l_1 m') \wedge \overline{\pi}(g_2' l_2 m') \wedge \overline{\pi}(g_1' l_1 n') \wedge \overline{\pi}(g_3' l_3 n') \to m' \perp n'$

O 11 $\overline{\pi}(abc) \wedge \overline{\pi}(abd) \wedge a \perp b \wedge c \perp d \to (\overline{\pi}(abx) \leftrightarrow \overline{\pi}(cdx))$

O 12 $\pi(abc) \wedge \gamma(abctlt'w'e) \wedge \overline{\pi}(abd) \wedge d \perp e \to \zeta(t'dw'e)$

O 13 $\pi(abc) \wedge \gamma(abctlt'w'e) \wedge \overline{\pi}(abd) \wedge d \perp e \wedge \varphi(abcmn) \to \overline{\pi}(mnd)$

O 14 $\pi(abc) \wedge \overline{\pi}(abd) \wedge \gamma(abctlt'w'e) \wedge \gamma(abcxjx'z'e') \wedge d \perp e \to d \perp e'$

O 15 $(\forall abc)(\exists dtlt'w'e)\, \pi(abc) \to \overline{\pi}(abd) \wedge \gamma(abctlt'w'e) \wedge d \perp e$

O 16 $(\forall abca'a''l)(\exists d)\, (l \perp a, b, c) \wedge \neq (abc) \wedge \zeta(aba'l) \wedge \zeta(a'ca''l) \to \zeta(ada''l)$

O 17 $(l \perp a, b, c) \wedge \neq (abc) \wedge \zeta(aba'l) \wedge \zeta(a'ca''l) \wedge \zeta(ada''l) \wedge w \perp l \wedge \overline{\zeta}(waxl)$
$\wedge \overline{\zeta}(xbyl) \wedge \overline{\zeta}(yczl) \to \overline{\zeta}(wdzl)$

O 18 $(l \perp a, b, c) \wedge \neq (abc) \wedge \gamma(abctmt'w'e) \wedge \zeta(aba'l) \wedge \zeta(a'ca''l) \wedge \zeta(ada''l) \to \zeta(t'dw'e)$

O 19 $\zeta(abcl) \wedge \gamma(abctmt'w'e) \to \zeta(t'bw'e)$

O 20 $(\exists abmnpqr)\, a \neq b \wedge (m \perp a, n) \wedge \pi(bmn) \wedge p \perp a \wedge q \perp b \wedge \pi(npq) \wedge m \neq p \wedge r \perp n \wedge \pi(brq) \wedge \pi(arp)$

O 21 $(\forall p_1 p_2 p_3 l)(\exists u)\, \pi(p_1 p_2 p_3) \to (\bigvee_{i=1}^{3}((u \perp p_i, l) \vee \pi(p_i l u)))$

O 22 $(\forall abl)(\exists c)\, (l \perp a, b) \wedge a \neq b \to \zeta(acbl)$

Notice that structures satisfying the above axioms must be non-elliptic metric planes, which can be expressed in terms of *points, lines, point-line incidence, line-orthogonality*, and *line-reflections* — defined as bijections of the collection of all points and lines, which preserve incidence and orthogonality, are involutory, and fix all the points of a line — by means of the following axioms:

MP 1 *There are at least two points.*

MP 2 *For every two different points there is exactly one line incident with those points.*

MP 3 *If a is orthogonal to b, then b is orthogonal to a.*

MP 4 *Orthogonal lines intersect.*

MP 5 *Through every point there is to every line a unique perpendicular.*

MP 6 *To every line there is at least a reflection in that line.*

MP 7 *The composition of reflections in three lines a, b, c which have a point or a perpendicular in common is a reflection in a line d.*

To see that the models of the axioms O1-O22 are metric planes, we use the interpretation of pairs of orthogonal lines as points together with the corresponding notion of point equality defined earlier, and define, for any line h and point $P = (a, b)$, the reflection of P in h to be the point (l, g_2) such that $\overline{\zeta}(g_1 h g_2 l)$, where l and g_2 are defined by $\overline{\pi}(abl) \wedge l \perp h$ and $\overline{\pi}(abg_1) \wedge g_1 \perp l$. By O1, O3-O6 this is a well-defined function, which, by O11, is independent of the particular choice of lines (a, b) representing the point P.

The axioms make the following statements: O1: orthogonality is symmetric; O2: there is no rectangle (this axiom's function is to ensure that, in case a and b are two lines with a common perpendicular l, the line PP' joining P, which is not incident with l, with its reflection P' in a, is different from line $P'P''$, which joins P' with its reflection P'' in b; this property is needed in O19 and O20, without it, we wouldn't know whether the γs in their antecedents ever hold[1]; O3 states the existence of a reflection of the point (l, g_1) in the line h whenever the point does not lie on h and O4 states the uniqueness of the reflection; O5 states the existence of a perpendicular to a given line e passing through the point (a, b) (or concurrent with or equal to one of two lines (a, b) for which there is a c with $\pi(abc)$) and O6 states the uniqueness of that perpendicular; O7 states the existence of a line m joining two different points (a, b) and (c, d) and O8 states its uniqueness; O9 states that reflections in h preserve collinearity and O10 states that they preserve orthogonality; O11: if two orthogonal lines c and d pass through (a, b), then a line x passes through (a, b) if and only if it passes through (c, d); O12 and O13: if the lines a, b, and c are concurrent and the image $R = (w', e)$ of the point $P = (l, t)$, with $l \perp a$, under $\sigma_c \sigma_b \sigma_a$ is different from P itself, then the perpendicular d, which is concurrent with (or equal to one of) the

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[1] That this axiom may be redundant, as may be the case with other axioms, doesn't quite matter, since we are not concerned with the independence of the axiom system, but only with the $\forall\exists$ nature of its axioms.

lines a and b, to the line $e = PR$ is also the perpendicular bisector of the segment having these two points as endpoints; and if (m, n) is a fixed point of $\sigma_c\sigma_b\sigma_a$, then it lies on d; O14: under the same hypotheses as above, the line d is the perpendicular bisector of *any* segment joining a point and its image under $\sigma_c\sigma_b\sigma_a$; O15: if a, b, and c and three concurrent lines, then there exists a point $P = (l, t)$ which is not a fixed point of $\sigma_c\sigma_b\sigma_a$ and there exists a line d concurrent with (or equal to one of) a and b, which is perpendicular to the line joining P with its image under $\sigma_c\sigma_b\sigma_a$; O16: if a, b, and c are three different lines, which have a common perpendicular l, such that c is not the reflection of a in b, and a'' is the image under $\sigma_c\sigma_b$ of a, then the perpendicular bisector d of the segment whose endpoints are (a, l) and (a'', l) exists; O17 and O18: if a, c, and c are three different lines, which have a common perpendicular l, such that c is not the reflection of a in b, w is any line perpendicular to l, and z is the image of w under $\sigma_c\sigma_b\sigma_a$, then z is the image of w under σ_d, where d is the line stated to exist in O16; and if $P = (t, m)$ is not equal to its image R under $\sigma_c\sigma_b\sigma_a$, then d is the perpendicular bisector of PR; O19: if c is the reflection of a in b, and l is the common perpendicular of these three lines, $P = (t, m)$ is a point which does not lie on l, and $R = (w', e)$ is the image of P under $\sigma_c\sigma_b\sigma_a$, then R is the reflection of P in b: O20: a and b are limiting parallel lines according to BERGAU's parallelism criterion; O21: given a line l and three concurrent lines p_1, p_2, p_3, we can find a line u which is either a common perpendicular of p_i and l, or which is concurrent with p_i and l, for some $i \in \{1, 2, 3\}$; O22: for any two lines a and b which have a common perpendicular l, there is a line c, such that b is the reflection of a in c.

It is now plain that with the notion of point and point-line incidence defined above, models of O1-O21 are models of KLINGENBERG's generalized hyperbolic planes, and those of O1-O22 of hyperbolic planes.

Notice that all the axioms, when written in our official language, without the use of any abbreviation, are ∀∃-sentences, as all the defined notions appearing in them have only existential quantifiers in their definiens, which turn into universal quantifiers whenever that notion appears in the antecedent of an axiom, and which appear *after* all universal quantifiers whenever it appears in the succedent of an axiom. Thus

Theorem 2. {O1-O21} *is a ∀∃-axiom system for* KLINGENBERG*'s generalized hyperbolic planes. Adding* O22 *we obtain a ∀∃-axiom system for hyperbolic planes over Euclidean ordered fields.*

4 Higher Dimensions

To get a ∀∃-axiomatization in $\mathcal{L}$ for n-dimensional hyperbolic geometry with $n \geq 3$ over Euclidean ordered fields, we need to:
(i) add the axiom

$$\neg L(abc) \wedge P(abcx) \wedge P(abcy) \wedge L(xyz) \wedge x \neq y \rightarrow P(abcz)$$

where $P(abcx) :\Leftrightarrow (\exists b'c')\, L(abb') \wedge L(acc') \wedge a \neq b' \wedge a \neq c' \wedge L(b'c'x)$, which ensures that x belongs to the plane spanned by three non-collinear points a, b, c if and only if $P(abcx)$;

(ii) add to the hypothesis of those axioms, for which it is possible that the points involved are not in same plane, conditions that ensure that all points are in the same plane. For example, for A16, we need to add $P(oabc)$ and $P(oabx)$ to the antecedent (hypothesis) to ensure that the points involved are all in one plane. This introduces existential quantifiers only in A21, which is a purely existential statement. All other axioms that require these added hypotheses stay purely universal;
(iii) add lower- and upper-dimension axioms, the lower one stating that there are o, x_i, u_{ij} with $i, j \in \{1, 2, \ldots, n\}$ such that $\bigwedge_{1\leq i<j\leq n}(x_io \perp_{u_{ij}} ox_j \wedge \neg L(ox_ix_j))$, and the upper one denying this statement with $i, j \in \{1, 2, \ldots, n+1\}$.
(iv) With $M(xyz)$ standing for $\sigma(yx) = z$, and $S_u(abc)$ standing for $a = c \vee b = c \vee (\neg L(abc) \wedge ac \perp_u cb)$, add the following axiom: If $a \neq b$, $M(akc)$, $M(\sigma(kb)sd)$, $S_u(cds)$, $M(ale)$, $M(\sigma(lb)mf)$, $S_v(efm)$, $M(cne)$, $M(\sigma(nd)pf)$, $M(\sigma(fe)q\sigma(\sigma(nd)e))$, then $e = p$ or $f = \sigma(nd)$ or $L(epq) \wedge \neg L(epf)$.
This axiom is precisely the statement that the congruence relation $\equiv$, defined by

$$ab \equiv cd :\Leftrightarrow (\exists klu)\, M(akc) \wedge M(\sigma(kb)ld) \wedge S_u(cdl),$$

is transitive, i. e. that $ab \equiv cd \wedge ab \equiv ef \to cd \equiv ef$.
Given that we can define an order relation for three collinear points by means of (cf. [24])

$$B(abc) :\Leftrightarrow (\exists duv)\, L(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a \wedge ad \perp_u dc \wedge db \perp_v ac \wedge \neg L(acd)$$

which says that b is between a and c if and only if it is the footpoint of the altitude to the hypotenuse in a right triangle, and this order satisfies all the usual properties required of it, it now follows from the theorem proved in [8] that the axiom system described above is an axiom system for n-dimensional hyperbolic spaces over Euclidean fields.

Similar adjustments are needed for producing an axiom system in terms of lines and $\perp$ for all dimensions $n \geq 4$, for which the notion of point-line incidence $\overline{\pi}'$, with points defined in the same manner as in the 2-dimensional case as pairs of perpendicular lines, can be defined as in [16, p. 58].

We first show how we can express in a purely existential manner the fact that a point (g, h) lies in the plane determined by two orthogonal lines a and b. Notice that the notions π' and $\overline{\pi}'$, with $\pi'(abl)$ and $\overline{\pi}'(abl)$ standing for *l, a line different from both a and b, goes through the intersection point of two perpendicular lines a and b*, and *a and b are orthogonal and l passes through their intersection point*, is defined in [16, p. 58] purely existentially as well. For improved readability, we shall use the defined notion χ, with $\chi(gh) :\Leftrightarrow (\exists a)\, a \perp g \wedge \overline{\pi}'(agh)$, to be read as *$g$ intersects h*.

Let $\xi(abmnpqr)$ denote the sentence in O20 without the quantifier prefix. To state that the point (g, h) lies in the plane determined by a and b with $a \perp b$, we shall ask that there is a line u passing through (g, h), which intersects a and b in points different from the intersection point of a and b. Since we are not allowed to use the universal quantifier, we cannot state that the points are different, as this would involve $\neg\overline{\pi}'$, thus introducing universal quantifiers. What we shall do instead is to state that through the point of intersection of the line u with b there exists a line c such that a and c are hyperbolically (or limiting) parallel, i. e. such that $(\exists mnpqrv)\, \xi(acmnpqr) \wedge \pi'(bqv) \wedge \pi'(amv) \wedge \chi(vn)$, a formula which will be denoted by $\lambda(ac)$. Here we have added to $\xi(abmnpqr)$ the condition that there exists a line v which passes through the intersection points of m and a, and of q and b, and intersects m. This ensures that all the lines involved are in the plane determined by the two orthogonal lines n and r.

We can now state that the point (g, h) lies in the plane determined by a and b with $a \perp b$ by stipulating that

$$(\exists uc)\,(\chi(au) \wedge \chi(bu) \wedge \pi'(buc) \wedge \lambda(ac)) \vee (\overline{\pi}'(abg) \wedge \overline{\pi}'(abh)) \vee \overline{\pi}'(gha),$$

a formula we shall denote by $\Pi(abgh)$.

We can now define the notion π, to be read $\pi(abc)$ if and only if a, b, c are three different lines that line in the same plane and are concurrent, in terms of π' by

$$\pi(abc) :\Leftrightarrow (\exists uvb_1b_2c_1c_2) \neq (abc) \wedge a \perp u \wedge \bigwedge_{i+1}^{2} (b \perp b_i \wedge c \perp c_i \wedge \Pi(aubb_i) \wedge \Pi(aucc_i)) \wedge \overline{\pi}'(aub) \wedge \overline{\pi}'(auc)$$

We then need to make sure that in the definitions of ϱ and γ all the lines that need to be coplanar are so, and that in every axiom in which the antecedent (hypothesis) needs to have all lines coplanar, we add conditions ensuring that. This is no longer a problem, for given two orthogonal lines a and b (in case they don't already exist in the antecedent of an axiom, we can always add a line orthogonal to one of the already existing ones), to say that a line l is in the plane they determine is to say that two points on l lie in that plane. Thus l is in the plane determined by a and b with $a \perp b$ if and only if

$$(\exists ghm)\,(g, h \perp l) \wedge \pi'(lgm) \wedge \lambda(mh) \wedge \Pi(ablg) \wedge \Pi(ablh).$$

This allows the $\forall\exists$-formulation of an axiom system for hyperbolic planes inside the space of dimension ≥ 4. By adding appropriate dimension axioms, as well as an axiom expressing (iv) in our language (this is possible, since we can define segment congruence for points that are pairs of perpendicular lines (given that the 2-dimensional variant of ϱ and ζ allows the definition of the notion of a midpoint)), we obtain an axiom system for n-dimensional hyperbolic space with $n \geq 4$.

5 Concluding Remarks

The $\mathcal{L}$-axiomatization could have been phrased in a language with the ternary midpoint relation M or with the binary midpoint operation μ as well, and the axiom system would have remained $\forall\exists$. This is in stark contrast to Euclidean geometry, whose M-theory is very weak (cf. [23]). Most axioms being universal statements, we notice that the universal σ-theory of hyperbolic planes is very rich, and ask the following questions: Is the universal σ-theory (the set of all universal sentences formulated in $\mathcal{L}$, i. e. by means of σ alone) of hyperbolic planes finitely axiomatizable? How about the universal theory of both operations μ and σ? The same question asked for Euclidean planes produces a very weak theory, namely the theory axiomatized by means of the three axioms for σ and μ from [25], as well as axioms stating that the characteristic is 0. What are the hyperbolic equational theories of σ, of μ, and of σ and μ?

I use this opportunity to point out that the axiom (C5′) on p. 107 of my paper "On the planarity of the equilateral, isogonal pentagon", *Math. Pannon.* **14** (2003), 101-112, could be weakened by asking not that lines $\overline{MN}$ and $\overline{M'N'}$ meet in a point P, but only that they are coplanar, as this was all that was used in the proof of Theorem 2.

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