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Groups and Plane Geometry

Abstract. We show that the first-order theory of a large class of plane geometries and the first-order theory of their groups of motions, understood both as groups with a unary predicate singling out line-reflections, and as groups acting on sets, are mutually interpretable.

Keywords: Erlanger Programm, mutually interpretable theories, groups generated by involutions, group actions, line reflections, metric planes.

1. Introduction

Klein's Erlanger Programm established a connection between geometries and their groups of transformations and gave rise to the belief that the group somehow carries the same information as the geometry itself. This is by no means the case with arbitrary structures, which can even be rigid, i. e. have trivial automorphism group. The extent to which it is possible, for complete theories satisfying certain properties, to recover the structure of a model from its automorphism group has been analysed in [2]. The belief that knowledge of the group of automorphisms enables one to recover the geometry that group emerged from is undoubtedly strengthened by the size of the automorphism groups geometries usually have.

We shall establish in this paper that, in a very precise sense, for the class of Bachmann's non-elliptic metric planes — which is a class significantly larger than that of plane absolute geometry — the first-order theories of the group of automorphisms, of the underlying geometric structure, and of the group of automorphisms, as a group acting on the set of points of the geometric structure, are mutually interpretable.

It was shown in [12] that, in the Euclidean case over real closed fields, the theories of the group of automorphisms and of the underlying geometry are mutually interpretable. The present paper can be seen as an extension of that result to both a wider class of geometries and a larger class of theories under comparison. Similar investigations on the mutual interpretability of

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the group of automorphisms and the corresponding geometric structure can be found in [9] and [10].

2. Non-elliptic metric planes in group-theoretic presentation

We shall first present non-elliptic metric planes as they appear in [1]. Our language will be a one-sorted one, with variables to be interpreted as 'rigid motions', containing a unary predicate symbol G, with $G(x)$ to be interpreted as 'x is a line-reflection', a constant symbol 1, to be interpreted as 'the identity', and a binary operation $\circ$, with $\circ(a,b)$, which we shall write as $a \circ b$, to be interpreted as 'the composition of a with b'.

To improve the readability of the axioms, we introduce the following abbreviations:

$$
\begin{aligned}
a^2 &:\Leftrightarrow a \circ a, \\
\iota(g) &:\Leftrightarrow g \neq 1 \wedge g^2 = 1, \\
a|b &:\Leftrightarrow G(a) \wedge G(b) \wedge \iota(a \circ b), \\
J(abc) &:\Leftrightarrow \iota((a \circ b) \circ c), \\
pq|a &:\Leftrightarrow p|q \wedge G(a) \wedge J(pqa).
\end{aligned}
$$

The axioms are (we omit universal quantifiers whenever the axioms are universal sentences):

B 1. $(a \circ b) \circ c = a \circ (b \circ c)$,

B 2. $(\forall a)(\exists b)\, b \circ a = 1$,

B 3. $1 \circ a = a$,

B 4. $G(a) \rightarrow \iota(a)$,

B 5. $G(a) \wedge G(b) \rightarrow G(a \circ (b \circ a))$,

B 6. $(\forall abcd)(\exists g)\, a|b \wedge c|d \rightarrow G(g) \wedge J(abg) \wedge J(cdg)$,

B 7. $ab|g \wedge cd|g \wedge ab|h \wedge cd|h \rightarrow (g = h \vee a \circ b = c \circ d)$,

B 8. $\bigwedge_{i=1}^{3} pq|a_i \rightarrow G(a_1 \circ (a_2 \circ a_3))$,

B 9. $\bigwedge_{i=1}^{3} g|a_i \rightarrow G(a_1 \circ (a_2 \circ a_3))$,

B 10. $(\exists ghj)\, g|h \wedge G(j) \wedge \neg j|g \wedge \neg j|h \wedge \neg J(jgh)$, B 11. $(\forall x)(\exists ghj)\, G(g) \wedge G(h) \wedge G(j) \wedge (x = g \circ h \vee x = g \circ (h \circ j))$,

B 12. $G(a) \wedge G(b) \wedge G(c) \rightarrow a \circ (b \circ c) \neq 1$.

Since $a \circ b$ with $a|b$ represents a point-reflection, we may think of an unordered pair (a, b) with $a|b$ as a *point*, an element a with $G(a)$ as a *line*, two lines a and b for which $a|b$ as a pair of perpendicular lines, and say that a point (p, q) is *incident* with the line a if $pq|a$. With these figures of speech in mind, the above axioms make the following statements: B1, B2, and B3 are the group axioms for the operation $\circ$; B4 states that line-reflections are involutions; B5 states the invariance of the set of line-reflections, B6 states that any two points can be joined by a line, which is unique according to B7 (we shall denote the line joining the points (a, b) and (c, d) by $\langle (a, b), (c, d)\rangle$); B8 and B9 state that the composition of three reflections in lines that have a common point or a common perpendicular is a line-reflection; B10 states that there are three lines g, h, j such that g are h are perpendicular, but j is perpendicular to neither g nor h, nor does it go through the intersection point of g and h; B11 states that every motion is the composition of two or three line-reflections, and B12 states that the composition of three line-reflections is never the identity. The function of the last axiom, B12, is to exclude elliptic geometries, and thus to ensure that the perpendicular from a point not on a line to that line is unique. The theory of non-elliptic metric planes, axiomatized by {B1-B12}, will be denoted by $\mathcal{B}$.

According to [1, §3,4], there is, for every point (a, b) and line l, a unique line h through (a, b) which is perpendicular to l (i. e. such that $ab|h$ and $l|h$), which will be denoted by $(a, b) \perp l$.

3. Non-elliptic metric planes in terms of group actions

Another language in which one can express a theory equivalent to that of group-theoretically expressed non-elliptic metric planes is a two-sorted language, with uppercase variables for *points* and lowercase variables for *rigid motions*, and a binary operation $\cdot$, the first argument of which is a rigid motion, the second argument a point, and whose value is a point. The intended interpretation of $\cdot(g, A)$, which will be written as $g \cdot A$, is 'the action of g on A' (see e. g. [5] for geometries defined by groups acting on sets).

Such an axiomatization was presented in [3].

For improved readability of the axioms we shall use the following abbreviations: 1 stands for the motion which satisfies

$$(\forall P)\, 1 \cdot P = P;$$

$a \circ b$ stands for the motion satisfying

$$(\forall P)\,(a \circ b) \cdot P = a \cdot (b \cdot P);$$

any equality $a = b$ or negated equality $a \neq b$ between motions, in which $\circ$ or 1 appears in a or b is understood as standing for $(\forall P)\, a \cdot P = b \cdot P$, or, in the negated case, for $(\exists P)\, a \cdot P \neq b \cdot P$. Since we use the abbreviation $\circ$, we may also use ι. We now define unary predicates characterizing rigid motions as proper rotation (ϱ), point-reflections (π), and line reflections (σ).

$$\begin{aligned} \varrho(a) \quad &:\Leftrightarrow \quad (\exists P)(\forall P')\,(a \cdot P = P \wedge (a \cdot P' = P' \rightarrow P' = P)), \\ \pi(a) \quad &:\Leftrightarrow \quad \varrho(a) \wedge \iota(a), \\ \sigma(a) \quad &:\Leftrightarrow \quad (\exists PQ)\, a \neq 1 \wedge P \neq Q \wedge a \cdot P = P \wedge a \cdot Q = Q. \end{aligned}$$

The axioms are:

M 1. $(\forall ab)(\exists P)\, a \cdot P \neq b \cdot P \vee a = b$,

M 2. $(\forall ab)(\exists c)\, a \circ b = c$,

M 3. $(\forall a)(\exists b)\, a \circ b = 1$,

M 4. $(\exists AB)\, A \neq B$,

M 5. $(\forall AB)(\exists g)\, A \neq B \rightarrow g \neq 1 \wedge g \cdot A = A \wedge g \cdot B = B$,

M 6. $\sigma(a) \wedge \sigma(b) \rightarrow \neg\sigma(a \circ b)$,

M 7. $\varrho(a) \wedge \varrho(b) \rightarrow \neg\sigma(a \circ b)$,

M 8. $(\forall a)(\exists P)\, \iota(a) \rightarrow a \cdot P = P$,

M 9. $\pi(a) \wedge \pi(b) \rightarrow \neg\pi(a \circ b)$,

M 10. $(\forall aAB)(\exists bP)\, a \cdot A = B \rightarrow b \cdot P = P \wedge b \cdot A = B$,

M 11. $\sigma(a) \wedge \sigma(b) \wedge \sigma(c) \rightarrow \neg\pi((a \circ b) \circ c)$.

Axiom M1 states that two motions a and b, with $a \cdot P = b \cdot P$ for all P, must coincide, M2 states that there is a motion c whose action is that of $a \circ b$, M3 states the existence of an inverse for any given rigid motion, M4 that there are two different points, M5 that for any pair of different points there is a rigid motion which fixes both but is not the identity, M6 and M7 that the composition of two line-reflections or proper rotations is never a line-reflection, M8 that any involutory rigid motion has a fixpoint, M9 that the composition of two point-reflections is never a point-reflection, M10 that if A can be mapped into B by means of a rigid motion, then this can also be accomplished by means of a rigid motion which has a fixpoint, and M11 that the composition of three line-reflections is never a point-reflection.

The theory axiomatized by M1–M11 will be denoted by $\mathcal{M}$.

4. Non-elliptic metric planes as a theory of geometric constructions

Non-elliptic metric planes can also be axiomatized in the traditions of both synthetic geometry and geometric constructions.

The language contains only one sort of individual variables, to be interpreted as *points*, and two ternary operation symbols, F and π. $F(abc)$ is the foot of the perpendicular from c to the line ab, if $a \neq b$, and a itself if $a = b$, and $\pi(abc)$ is the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, or $a = b = c$, and arbitrary otherwise. By 'fourth reflection point' we mean the following: if we designate by σ_x the mapping defined by $\sigma_x(y) = \sigma(xy)$, where $\sigma(xy)$ stands for the reflection of y in the point x, then, if a, b, c are three collinear points, by [1, S3,9, Satz 24b], the composition $\sigma_c\sigma_b\sigma_a$, is the reflection in a point, which lies on the same line as a, b, c. That point is called the fourth reflection point corresponding to the triple (a, b, c), and is denoted by $\pi(abc)$.

For improved readability we shall use the following abbreviations:

$$\begin{aligned} \sigma(ab) &:= \pi(aba), \\ R(abc) &:= \sigma(F(abc)c), \\ L(abc) &:\leftrightarrow F(abc) = c \vee a = b, \end{aligned}$$

where σ has the same meaning as above, $R(abc)$ stands for the reflection of c in ab (a line if $a \neq b$, the point a if $a = b$), and $L(abc)$ stands for 'the points a, b, c are collinear (but not necessarily distinct)'.

The axiom system consists of the following axioms:

C 1. $F(aab) = a$,

C 2. $\sigma(aa) = a$,

C 3. $\sigma(a\sigma(ab)) = b$,

C 4. $\sigma(ax) = \sigma(bx) \rightarrow a = b$,

C 5. $L(aba)$,

C 6. $L(abc) \rightarrow L(cba) \wedge L(bac)$,

C 7. $L(ab\sigma(ab))$,

C 8. $L(abF(abc))$,

C 9. $a \neq b \wedge F(abx) = F(aby) \rightarrow L(xyF(abx))$, C 10. $a \neq b \wedge c \neq d \wedge F(abc) = c \wedge F(abd) = d \rightarrow F(abx) = F(cdx)$,

C 11. $\neg L(abx) \wedge F(xF(abx)y) = y \rightarrow F(abx) = F(aby)$,

C 12. $a \neq b \wedge a \neq c \wedge F(abc) = a \rightarrow F(acb) = a$,

C 13. $a \neq x \wedge x \neq y \wedge F(axy) = x \rightarrow F(a\sigma(ax)\sigma(ay)) = \sigma(ax)$,

C 14. $\sigma(\sigma(xa)\sigma(xb)) = \sigma(x\sigma(ab))$,

C 15. $u \neq v \wedge a \neq b \wedge F(abc) = a \rightarrow F(R(uva)R(uvb)R(uvc)) = R(uva)$,

C 16. $\neg L(oab) \wedge \neg L(obc) \rightarrow \sigma(F(xR(ocR(obR(oax)))o)x)$
$= R(ocR(obR(oax)))$,

C 17. $\neg L(oab) \wedge \neg L(obc) \wedge \sigma(mx) = R(ocR(obR(oax)))$
$\wedge \sigma(ny) = R(ocR(obR(oay))) \rightarrow L(omn)$,

C 18. $a \neq b \wedge b \neq c \wedge F(abc) = c \wedge a \neq a' \wedge b \neq b' \wedge c \neq c'$
$\wedge F(aba') = a \wedge F(bab') = b \wedge F(cbc') = c$
$\rightarrow \sigma(F(xR(cc'R(bb'R(aa'x)))\pi(abc))x) = R(cc'R(bb'R(aa'x)))$
$\wedge F(\pi(abc)cF(xR(cc'R(bb'R(aa'x)))\pi(abc))) = \pi(abc)$,

C 19. $(\exists abc)\neg L(abc)$.

The axioms make the following statements: C1 defines the value of $F(abc)$ when $a = b$ — it is an axiom with no geometric function (we could have opted to leave it undefined, but that would have lengthened the statements of the axioms C16 and C18); C2: the point a is a fixed point of the reflection σ_a, C3: reflections in points are involutory transformations (or the identity); C4: reflections of a point in two different points do not coincide; C5: a lies on the line determined by a and b; C6: collinearity of three points is a symmetric relation; C7: the reflection of b in a is collinear with a and b; C8: for $a \neq b$, the foot of the perpendicular from c to the line ab lies on that line; C9 states the uniqueness of the perpendicular to the line ab in the point $F(abx)$; C10: the foot of the perpendicular from x to the line ab does not depend on the particular choice of points a and b that determine the line ab; C11: if x is a point outside of the line ab, and y is a point on the perpendicular from x to ab, then the feet of the perpendiculars of x and y to the line ab coincide; C12 states that perpendicularity is a symmetric relation (if ca is perpendicular to ab, then ba is perpendicular to ac); C13: if yx is perpendicular to xa, the so are $\sigma_a(y)\sigma_a(x)$ and $\sigma_a(x)a$; C14 states a certain preservation of the operation σ under reflections in points; C15: reflections in lines preserve the orthogonality relation; C16 and C17 together state the three reflections theorem for lines having a point in common; C18 is the three reflections theorem for lines having a common perpendicular; C19: there are three non-collinear points.

The theory axiomatized by C1-C19 will be denoted by $\mathcal{C}$.

Another first-order axiomatization for non-elliptic metric planes, with one sort of variables for *lines*, and a binary operation ϱ, with $\varrho(a,b)$ to be interpreted as 'the reflection of line b in line a', which follows very closely Bachmann's axiom system for $\mathcal{B}$ has been provided in [8].

5. Mutual interpretability

To show that the theories $\mathcal{B}$, $\mathcal{M}$, and $\mathcal{C}$ are mutually interpretable, we shall use the notion of mutual interpretability for theories with different intended interpretations of the individual variables presented in [13], [6], [11].

To translate from $\mathcal{C}$ to $\mathcal{B}$, we define a *point* to a be a pair (g,h) of line-reflections satisfying $g|h$, and we define an equivalence notion among such pairs (to be treated as point-equality) by

$$(g,h) \equiv (g',h') :\Leftrightarrow g \circ h = g' \circ h'$$

The operations F and π are defined on these pairs of line-reflections as follows (the definition given below for F is valid only for $(a,a') \not\equiv (b,b')$; in case $(a,a') \equiv (b,b')$, $F((a,a'),(b,b'),(c,c'))$ is defined as (a,a')):

$$\begin{aligned} F((a,a'),(b,b'),(c,c')) : \ &\Leftrightarrow \ (\langle (a,a'),(b,b')\rangle, (c,c') \perp \langle (a,a'),(b,b')\rangle), \\ \pi((g,a),(g,b),(g,c)) : \ &\Leftrightarrow \ (g, c \circ (b \circ a)). \end{aligned}$$

It is easy to verify that with these definitions, F and π do satisfy the axioms of $\mathcal{C}$.

To translate in the opposite direction, from $\mathcal{B}$ to $\mathcal{C}$, we first introduce some abbreviations: $I(xyz) :\Leftrightarrow y = z \vee \sigma(F(yzx)y) = z$ stands for 'xy is congruent to xz'. We define, for $\sigma(mb) \neq b'$, $n := F(b'\sigma(mb)a')$ and

$$p := F(bb'\pi(mF(mn\sigma(mb))n)). \tag{1}$$

If $I(a'b'\sigma(mb))$, then n represents the midpoint of the segment $\sigma(mb)b'$, and p represents the midpoint of the segment bb'. That in a (possibly degenerate) triangle, in which two sides have midpoints, the third side has a midpoint as well, was proved in [1, §4,2, Satz 2 (Satz von der Mittellinie)], and it is the construction from that proof that we have used to obtain p.

We now define a *rigid motion* to be a 7-tuple (a, b, c, a', b', c', m) with

$$\neg L(abc) \wedge \sigma(ma) = a' \wedge I(a'b'\sigma(mb)) \wedge I(a'c'\sigma(mc)) \wedge ((\sigma(mb) \neq b' \\ \wedge I(b'c'\sigma(pc)) \vee (\sigma(mb) = b' \wedge I(b'c'\sigma(mc)))).$$

Thus a rigid motion is defined by its action on a triangle abc, and the definition states that $a'b'c'$ is a triangle that is congruent to abc, and that m is the midpoint of aa'. That the segment having as endpoints a point and its image under a rigid motion always has a midpoint in a metric plane was proved in [1, §3,10, Satz 28].

Let

$$\begin{aligned} \lambda(aba'b'xx') \quad :\Leftrightarrow \quad & (\sigma(mb) = b' \wedge F(a'b'x') = \sigma(mF(abx))) \vee (\sigma(mb) \neq b' \\ & \wedge F(a'b'x') = R(a'R(\sigma(mb)b'a')\sigma(mF(abx)))) \end{aligned}$$

stand for 'the distances to a and b of the foot $F(abx)$ of the perpendicular from x to the line ab are equal to the corresponding distances to a' and b' of the foot $F(a'b'x')$ of the perpendicular of x' to the line $a'b'$'.

We now introduce another abbreviation, $\kappa(abca'b'c'mxx')$, standing for 'x' is the image of x under the rigid motion (a, b, c, a', b', c', m)', defined by

$$\kappa(abca'b'c'mxx') :\Leftrightarrow \lambda(aba'b'xx') \wedge \lambda(aca'c'xx').$$

For every rigid motion (a, b, c, a', b', c', m) and every x there is exactly one x' such that $\kappa(abca'b'c'mxx')$ holds.

Two 7-tuples, $(a_i, b_i, c_i, a'_i, b'_i, c'_i, m_i)$ for $i = 1, 2$ represent the same rigid motion, i. e. $(a_1, b_1, c_1, a'_1, b'_1, c'_1, m_1) \equiv (a_2, b_2, c_2, a'_2, b'_2, c'_2, m_2)$ if

$$\kappa(a_1b_1c_1a'_1b'_1c'_1m_1a_2a'_2) \wedge \kappa(a_1b_1c_1a'_1b'_1c'_1m_1b_2b'_2) \wedge \kappa(a_1b_1c_1a'_1b'_1c'_1m_1c_2c'_2).$$

A rigid motion $r := (a, b, c, a', b', c', m)$ is a line-reflection, that is, $G(r)$ holds, if and only if

$$\begin{aligned} & (m \neq p \wedge F(mpc) = q \wedge F(mpa) = m \wedge F(mpb) = p) \\ & \vee (m \neq q \wedge F(mqc) = q \wedge F(mqa) = m \wedge F(mqb) = p) \\ & \vee (q \neq p \wedge F(pqc) = q \wedge F(pqa) = m \wedge F(pqb) = p), \end{aligned}$$

where p and q denote the midpoints of bb' and cc', which are defined in terms of a', b, b', c, c', m as done for p in (1). The definition states that (a, b, c, a', b', c', m) is a line-reflection if and only if the midpoints m, p, q of aa', bb', cc' are not all equal, and a', b', c' are reflections of a, b, c in the line determined by m, p, q.

We are now left with the task of defining $\circ$ for these 7-tuples. We start with two 7-tuples (a,b,c,a',b',c',m) and (a',b',c',x,y,z,n) and define

$$(a,b,c,a',b',c',m)\circ(a',b',c',x,y,z,n) := (a,b,c,x,y,z,p),$$

where p is the midpoint of ax, which exists by [1, §4,2, Satz 2], given that aa' and $a'x$ have midpoints, and can be determined as in (1). Now $(a,b,c,a',b',c',m)\circ(d,e,f,d',e',f',m')$ is defined to be $(a,b,c,a',b',c',m)\circ(a',b',c',x,y,z,n)$, where (a',b',c',x,y,z,n) is the 7-tuple for which $(d,e,f,d',e',f',m')\equiv(a',b',c',x,y,z,n)$. Here x,y,z were determined as the unique solutions to $\kappa(defd'e'f'm'a'x)$, $\kappa(defd'e'f'm'b'y)$, $\kappa(defd'e'f'm'c'z)$ and n is the midpoint of $a'x$.

That all axioms of $\mathcal{B}$ are satisfied with these definitions is now a matter of verification, for the most part accomplished in [7].

To translate from $\mathcal{M}$ to $\mathcal{B}$, we define points in the same manner as done in the translation from $\mathcal{C}$ to $\mathcal{B}$, and, for $P=(a,b)$ we define $g\cdot P$ as $(g\circ a\circ g^{-1}, g\circ b\circ g^{-1})$. The fact that all axioms of $\mathcal{M}$ are satisfied under this translation has been proved in [3].

To translate from $\mathcal{B}$ to $\mathcal{M}$ we need only define the meaning of G in $\mathcal{M}$ and the operation $\circ$ for the rigid motions of $\mathcal{M}$. The definitions are:

$$\begin{aligned} G(g) \quad &:\Leftrightarrow \quad (\exists ABP)\, A\neq B \wedge g\cdot A = A \wedge g\cdot B = B \wedge g\cdot P\neq P,\\ g\circ h = m \quad &:\Leftrightarrow \quad (\forall P)\, g\cdot(h\cdot P) = m\cdot P. \end{aligned}$$

That the axioms of $\mathcal{B}$ are satisfied under this translation was shown in [3].

6. Must line-reflections be singled out in $\mathcal{B}$?

To the group-theoretic purist Bachmann's axiom system for $\mathcal{B}$ suffers from a major deficiency: it is not expressed in purely group-theoretical terms, as it contains a unary predicate G for singling out line-reflections. Did we actually need G or could we have defined it in purely group-theoretical terms? The positive answer is given in [1, p. 235]. In hyperbolic motion groups it is not always possible to distinguish the two types of involutory elements, namely point- and line-reflections. Hyperbolic motion groups satisfy besides the axioms of $\mathcal{B}$ the axioms $\sim V^*$ and H from [1, §14,1], and they are isomorphic to the group $PGL(2,K)$ for some ordered field K. This group consists of 2×2 homogeneous matrices with nonzero determinant, and the involutory elements of the group are represented by traceless 2×2 homogeneous matrices with nonzero determinant. The line-reflections are precisely those involutory elements with negative determinant, and the point-reflections are those with positive determinant. Let $K = \mathbb{Q}(t)$, ordered such that t turns out to be 'infinitely large', i. e. by defining $(\Sigma_{i=1}^{m} a_i t^i)(\Sigma_{j=1}^{n} b_j t^j)^{-1}$ to be > 0 if and only if $a_m b_n > 0$. We define an automorphism of $PGL(2, K)$ by extending the mapping $t \mapsto t^{-1}$ to an automorphism of the field K and then to an automorphism φ of the group $PGL(2, K)$. The homogeneous matrix

$$r\begin{pmatrix} t & -1 \\ 1 & -t \end{pmatrix}$$

which represents a line-reflection, since an element of its determinant's quadratic class is $1 - t^2$, thus negative, gets mapped under φ into

$$r\begin{pmatrix} t^{-1} & -1 \\ 1 & -t^{-1} \end{pmatrix}$$

which represents a point-reflection, given that an element of its determinant's quadratic class is $1 - t^{-2}$, which is positive. By Padoa's method, G cannot be defined in terms of $\circ$ in the case of hyperbolic motion groups, and consequently G cannot be defined inside $\mathcal{B}$ either.

However, there are two special cases in which G can be defined in group-theoretic terms: the case in which the metric is Euclidean, that is, in which there exists a rectangle (axiom R in [1, §6,7]), i. e.

$$(\exists abcd)\, a \neq b \wedge c \neq d \wedge a|c \wedge a|d \wedge b|c \wedge b|d,$$

and the case in which for every line there exists a line which has neither a point nor a perpendicular in common with it. Given that, by [1, Satz 17], involutions can be only line- or point-reflections, we can express this axiom as follows

$$(\forall g)(\exists h)(\forall p)\, G(g) \rightarrow G(h) \wedge (\iota(p) \rightarrow \neg(\iota(p \circ g) \wedge \iota(p \circ h))). \tag{2}$$

Axiom (2) is satisfied by plane hyperbolic geometry coordinatized by Euclidean fields, but is weaker than the axiom H^* of [1, §15,2], which together with the axioms of $\mathcal{B}$ axiomatize plane hyperbolic geometry coordinatized by Euclidean fields.

In the presence of axiom R postulating a Euclidean metric, G can be defined by

$$\begin{aligned} G(g) \quad :\Leftrightarrow \quad & (\exists abc)\, \iota(g) \wedge \iota(a) \wedge \iota(b) \wedge \iota(c) \wedge a \neq b \wedge c \neq g \\ & \wedge \iota(g \circ a) \wedge \iota(g \circ b) \wedge \iota(c \circ a) \wedge \iota(c \circ b). \end{aligned}$$

This definition states that (i) g is an involution and that (ii) there are a, b, c, with $a \neq b$ and $c \neq g$, such that a and b are common points or perpendiculars to both c and g. If g were a point, then a and b would have to be lines, and g would have to be their intersection point. Since a and b cannot be perpendiculars from g to c, given the uniqueness of the perpendicular from a point to a line, c must be a point as well. However, that is not possible either, for the lines a and b cannot have two different points in common.

In the presence of axiom (2), G can be defined by

$$G(g) :\Leftrightarrow (\exists h)(\forall p)\, \iota(g) \wedge \iota(h) \wedge (\iota(p) \rightarrow \neg(\iota(p \circ g) \wedge \iota(p \circ h))).$$

This definition states that (i) g is an involution and that (ii) there exists an involution h, which has neither a common point nor a common perpendicular with g. If g were a point, then regardless of whether h is a point or a line, the definiens in the above definition would not be true, for if h were a point, then the joining line $\langle g, h\rangle$ would furnish a p with $\iota(p \circ g) \wedge \iota(p \circ h)$, and if h were a line, then $g \perp h$, the line through g perpendicular to h, would furnish such a p. Thus g must be a line.

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