SPLITTING THE PASCH AXIOM

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Pasch's axiom is shown to be equivalent to the conjunction of the following two axioms: "In any right triangle the hypotenuse is greater than the leg" and "If $\angle \mathsf{AOB}$ is right, B lies between O and C, and D is the footpoint of the perpendicular from B to AC, then the segment OA is greater than the segment BD."

1 INTRODUCTION

In [3] we have shown how the Euclidean parallel postulate can be split into two weaker geometrically meaningful axioms. We have also motivated the operation of splitting axioms in that paper and shall not repeat those arguments here. In this paper, we shall attempt to split the Pasch axiom.

The plane Euclidean geometry of ruler and gauge constructions, considered as a first-order theory in a language L with two relation symbols — $\equiv$ (quaternary) and B (ternary) — will be denoted by $\mathcal{E}$. Its models are Cartesian planes over Pythagorean ordered fields.

Let F be a formally real and Pythagorean field and $\leq$ an ordering of the additive group of F with $0 \leq 1$; $\leq$ will be called a normed semi-ordering of F. Let $P = \{x \in F : x \geq 0\}$ be the set of *semi-positive* elements of F, and let $\|\cdot\| : F \times F \to P$ be defined by $\|(x, y)\| = \sqrt{x^2 + y^2}$. Using $\|\cdot\|$, we can define a notion of congruence ($\equiv_F$) and betweenness ($\mathbf{B}_F$) by setting $\mathbf{ab} \equiv_F \mathbf{cd}$ iff $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|$ and $\mathbf{B}_F(\mathbf{abc})$ iff $\|\mathbf{a} - \mathbf{b}\| + \|\mathbf{b} - \mathbf{c}\| = \|\mathbf{a} - \mathbf{c}\|$.[1] The structure $\langle F \times F, \equiv_F, \mathbf{B}_F\rangle$ will be called a semi-ordered Cartesian plane.

Let $\mathcal{E}^-$ stand for 2-dimensional Pasch-free Euclidean geometry, the first-order theory, ex-

[1] Here $\mathbf{v} = (v_1, v_2)$ and $\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2)$.

pressed in L, whose models are semi-ordered Cartesian planes. This theory was introduced in L. W. SZCZERBA [6] and a representation theorem for it was proved in L. W. SZCZERBA and W. SZMIELEW [7]. All pure congruence-theorems from $\mathcal{E}$ are in $\mathcal{E}^-$ as well. The Pasch axiom (P), however, is not in $\mathcal{E}^-$; by adding P to $\mathcal{E}^-$ we get $\mathcal{E}$.

H. N. GUPTA and A. PRESTEL [2] have considered a weakening of P, which can be equivalently stated as either "The footpoint of the altitude of a right triangle lies between the endpoints of the hypotenuse" or "In every right triangle, the hypotenuse is greater than the legs" (R) or "The triangle inequality".[2] The equivalence of these statements, provable in $\mathcal{E}^-$, is shown in [4, Satz 2.3]. It was shown in [2] that the models of $\mathcal{E}^-$ and R are quadratically semi-ordered Cartesian planes, that is, the semi-order of the coordinate field F satisfies the condition

$$0 \leq x \rightarrow 0 \leq xy^2. \tag{1}$$

By constructing a quadratically semi-ordered formally real and Pythagorean field which is not ordered, GUPTA and PRESTEL [2] have shown that R is weaker than P, i. e.

$$\mathcal{E}^- \nvdash R \rightarrow P.$$

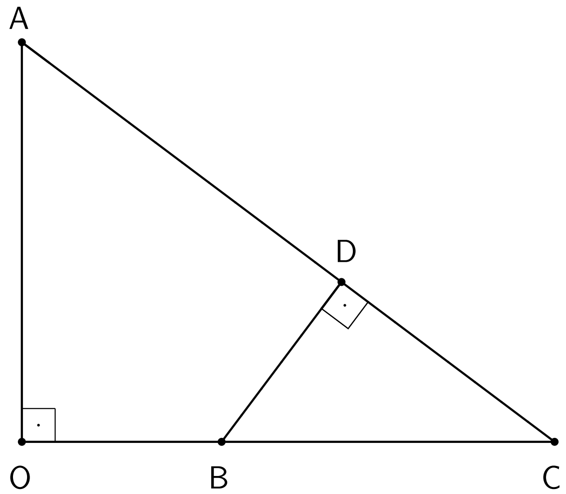
It is therefore natural to ask for the missing link from R to P, that is, for a geometrically meaningful statement R', such that

$$\mathcal{E}^- \vdash P \leftrightarrow R \wedge R', \tag{2}$$

$$\mathcal{E}^- \nvdash R' \rightarrow P, \tag{3}$$

$$\mathcal{E}^- \nvdash R \vee R'. \tag{4}$$

[2]We shall choose the statement R as the geometric counterpart of (1), since it requires a smaller number of variables in its formulation in the language L. It can be stated as $(\forall oabb'c)\, o \neq a \wedge o \neq b \wedge B(bob') \wedge ob \equiv ob' \wedge ab \equiv ab' \wedge ac \equiv ao \wedge (B(acb) \vee B(abc)) \rightarrow B(acb)$.



Axiom R' states that BD is shorter than OA

We shall prove that the statement (R'): "If $\angle$AOB is right, B lies between O and C (with B $\neq$ O and B $\neq$ C), and D is the footpoint of the perpendicular from B to AC, then the segment OA is greater than the segment BD" satisfies (2) and (4). We were unable to either prove or disprove (3).

2 THE PROOF

We shall first prove that R and R' satisfy (2). The $\rightarrow$-part of it is trivial, since it amounts to checking that R and R' are valid in Cartesian planes over Pythagorean ordered fields. To prove the converse, let $(F, \leq)$ be the coordinate field of a semi-ordered Cartesian plane satisfying R and R', and $x > 0, t > 0$ be any two non-zero semi-positive elements in F. With $\mathsf{O} = (0,0), \mathsf{A} = (0,1), \mathsf{B} = (x,0), \mathsf{C} = (x+t,0)$ in (R'), the statement "BD is shorter than OA" made by the axiom (R') becomes

$$\left| \frac{t}{\sqrt{1+(x+t)^2}} \right| < 1. \tag{5}$$

Since the semi-order is quadratic (as R is valid in our Cartesian plane as well), we can deduce from (5) that (cf. [5, Lemma (1.18)])

$$\frac{t^2}{1+(x+t)^2} < 1, \tag{6}$$

which, in turn, implies that

$$\frac{1+x^2+2xt}{1+(x+t)^2} > 0.$$

Multiplying by $1+(x+t)^2$, which is a square, we get

$$1 + x^2 + 2xt > 0. \tag{7}$$

We have thus established that R and R' imply that the quadratically semi-ordered coordinate field $(F, \leq)$ satifies (7) for all $x > 0, t > 0$ in F.

Suppose that, for some $x > 0$ and $t > 0$ in F, $xt < 0$. By (7) and the fact that the semi-order is quadratic, these particular values of x and t have to satisfy

$$1 + x^2 + 2xtq^2 > 0, \text{ for all } q \in F. \tag{8}$$

With $q = 1/t$, (8) becomes

$$-\frac{2x}{t} < 1 + x^2,$$

which implies (cf. [5, Lemma (1.18)])

$$-\frac{t}{2x} > \frac{1}{1+x^2}, \text{ and hence } -\frac{t(1+x^2)^2}{2x} > 1 + x^2,$$

which contradicts (8) with $q = \frac{1+x^2}{2x}$.

Therefore, for all $x > 0$ and $t > 0$ we must have $xt > 0$, i. e. the semi-order is an order. This proves (2).

To prove (4), we shall construct a semi-order on the field of real numbers $\mathbb{R}$ following the method used in [6]. Let

$$a = \sqrt{1 - \frac{1}{2\sqrt{2}}}.$$

Extend $\{1, a, a^2\}$ to a basis $\mathfrak{B}$ of $\mathbb{R}$ over $\mathbb{Q}$. Let φ be the linear automorphism defined on $\mathfrak{B}$ by $\varphi(a^2) = -a^2$ and $\varphi(b) = b$ for all $b \in \mathfrak{B} \setminus \{\mathfrak{a}^2\}$. Let $\varphi(P) = P_1$, where P is the positive cone of $\mathbb{R}$. Define now $\leq_1$ on $\mathbb{R}$ by $x \leq_1 y$ iff $y - x \in P_1$. $(\mathbb{R}, \leq_1)$ is a semi-ordered field which is not quadratically semi-ordered, since a^2, a square, is negative. $(\mathbb{R}, \leq_1)$ does not satisfy (5) with $x = t = 1/2$. Therefore neither R nor R' are valid in the semi-ordered Cartesian plane over $(\mathbb{R}, \leq_1)$. This proves (4).

Note that in the above construction we could have constructed the semi-order $\leq_1$ on the Euclidean closure $Eu(\mathbb{Q})$ of $\mathbb{Q}$ (that is, the smallest Euclidean subfield of $\mathbb{R}$). In this way we would have avoided the use of the Axiom of Choice in extending $\{1, a, a^2\}$ to a basis.

It is worth noting that, if we set $\mathsf{B} = \mathsf{O}$ in R', we get an axiom that is equivalent to R (cf.[4, Satz 2.3]). Therefore, if we omit the condition $\mathsf{B} \neq \mathsf{O}$ in the antecedent of R', we get an axiom P' that is equivalent — given $\mathcal{E}^-$ — to P.

It is also worth mentioning that L. M. Kelly's proof of Sylvester's problem (cf. [1, p. 65]) uses both and R and R'. According to (2), this means that, in the Euclidean case, his proof uses the full power of the ordering of the plane. It remains open whether this is true in the absolute case as well.

References

[1] H. S. M. Coxeter, *Introduction to geometry*, Wiley, New York, 1989.

[2] H. N. Gupta, A. Prestel, On a class of Pasch-free Euclidean planes, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys.* **20** (1972), 17-23.

[3] V. Pambuccian, Zum Stufenaufbau des Parallelenaxioms, *J. Geom.* **51** (1994), 79-88.

[4] A. Prestel, Euklidische Geometrie ohne das Axiom von Pasch, *Abh. Math. Sem. Univ. Hamburg* **41** (1974), 82-109.

[5] A. Prestel, *Lectures on formally real fields*, Lecture Notes in Mathematics, vol. 1093, Springer-Verlag, Berlin, 1983.

[6] L. W. Szczerba, Independence of Pasch's axiom, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **18** (1970), 491-498.

[7] L. W. Szczerba, W. Szmielew, On the Euclidean geometry without the Pasch axiom, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **18** (1970), 659-666.

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