

On the Planarity of the Equilateral, Isogonal Pentagon*

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Abstract: We analyse the assumptions of two proofs of the theorem stating that the vertices of pentagon with equal sides and equal diagonals in three dimensional space must lie in a plane, one of which is new, and the analysis of its assumptions show that this theorem remains valid in very general 3-dimensional geometries, such as regular affine metric geometries or rectangular spaces of characteristic $\neq 2, 5$, generalized hyperbolic spaces over arbitrary ordered fields, and double elliptic spaces.

1 Introduction

That a pentagon with equal sides and equal diagonals in \mathbb{R}^3 must be planar seems to have been rediscovered many times, and may have been known by Euler (see H. Martini [19] for a complete bibliography). Such pentagons will be referred to in the sequel as “regular pentagons”.

By analysing two proofs, one by Irminger [10] and one new, we will obtain two sets of weak assumptions under which the regular pentagon theorem, which will be referred to as PRP (as it asserts the Planarity of the Regular Pentagon) can be proved.

The tradition of systematic axiomatic analyses of theorems of sufficient interest goes back to at least 1936 ([3]), and includes [31], [32], [11], [2], [5], [8], [22], [23], [34], [35]. This enterprise may be called, borrowing a term from the current program of finding weak set

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existence axioms under which the theorems of classical mathematics remain true (cf. [29]), *reverse geometry*. Even if one is not interested in this type of analysis, one may find its fruits worth the effort, for such analyses often prove the validity of the analysed theorem in a larger class of geometries than the one in which it was originally discovered. That a generalization of PRP remains valid in the real hyperbolic and elliptic three dimensional spaces of PRP has been proved by Korchmáros [14], who provided a proof valid in all these geometries. We shall return to his proof in §4.

The necessity of one particular assumption for the validity of this theorem has already been established: that three-dimensionality is essential was pointed out by O. Bottema [4].

2 The First Proof

We shall first set up a very weak axiom system, in which Irmingier's [10] proof can be carried out. The first-order language in which the axiom system will be expressed contains two sorts of individual variables, for *points* and *planes*, to be denoted by Latin upper-case and Greek lower-case letters, and the binary predicate of point- plane incidence \in , the ternary predicate of collinearity Λ with points as arguments, and $\Lambda(A, B, C)$ to be read as ' A, B, C are collinear', the quaternary relation of equidistance \equiv with points as variables, where $AB \equiv CD$ is read 'segment AB is congruent to segment CD ', where by 'segment' we understand simply the point-pair formed by its endpoints, and Σ , the ternary relation of betweenness Σ with two point arguments and one plane argument, with $\Sigma(A, \pi, B)$ to be read as ' π lies between A and B ', and for which we write, for improved readability $A\pi B$. The reference for logical matters related to many-sorted languages is [21]. Given that our axiomatizations are elementary (first-order), no reference to set-theoretic notions is allowed anywhere in the axioms.

The first group of axioms are the incidence axioms:

- (I1) Collinearity is a symmetric relation between three distinct points, i. e. $\Lambda(A, B, C) \rightarrow A \neq B \wedge \Lambda(A, C, B) \wedge \Lambda(C, A, B)$.
- (I2) If A, B, C are collinear and A and B are in π , then C is incident with π as well.
- (I3) Every plane is incident with three non-collinear points.
- (I4) If A, B, C are three points, then there is a plane incident with them, and if the points are non-collinear the plane is unique, and will be denoted by $\pi(A, B, C)$.

The second group of axioms are congruence axioms:

- (C1) (i) If $AB \equiv CD$ then $CD \equiv AB$;
- (ii) If $AB \equiv CD$ then $AB \equiv DC$;
- (iii) If $AB \equiv CD$ and $CD \equiv CE$ then $AB \equiv CE$;
- (iv) If $AB \equiv CD$ and $CD \equiv AE$ then $AB \equiv AE$.
- (C2) For any two distinct points A and B , there is a plane μ , such that every point incident with μ is equidistant from A and B , and all points equidistant from A and B are incident

with μ .

The plane is called the *perpendicular bisector plane* of AB , and is denoted by $\mu(A, B)$ (notice that it follows from (I3), (I4), and (C2) that $\mu(A, B) = \mu(B, A)$).

(C3) If P is not in π , then there is a unique point $P' \neq P$ such that $PX \equiv P'X$ for all $X \in \pi$. The point P' is also not in π , and will be denoted by $s_\pi(P)$.

(C4) If A, B, C are not collinear, P does not lie in $\pi(A, B, C)$, $PA \equiv P'A$, $PB \equiv P'B$, $PC \equiv P'C$, then $P' = P$ or $P' = s_{\pi(A, B, C)}(P)$.

(C5) If P and Q are not on π , then $PQ \equiv s_\pi(P)s_\pi(Q)$.

(C6) If A and B are two different points incident with a plane π , and C is a point not incident with π , then $s_{\mu(A, B)}(C)$ is also not incident with π .

(C7) If π is a plane and P, Q are two points not on π , then $P, Q, s_\pi(P)$ and $s_\pi(Q)$ are coplanar.

(C8) If A, B, C are collinear and $AC \equiv BD$, $AB \equiv CD$, then B, C, D are collinear as well. The third group of axioms are betweenness axioms:

(B1) If P is not in π , then π does not lie between P and P .

(B2) If A_i (for $i = 1, \dots, 5$) are such that no four of them are contained in a plane, then there are indices i, j, k such that $\pi(A_i, A_j, A_k)$ lies between $s_{\pi(A_i, A_j, A_k)}(A_l)$ and A_m , where $\{m, l\} = \{1, 2, 3, 4, 5\} \setminus \{i, j, k\}$.

(B3) If A, B are incident with π , P and R are not incident with π , and $R\pi P$, then $R\pi s_{\mu(A, B)}(P)$.

Irminger's [10] proof starts with

Lemma. *If four vertices of a regular pentagon are coplanar, then the fifth vertex lies in the same plane as well.*

Proof. Notice that, for all points P and Q with $P \neq Q$ we have

$$s_{\mu(P, Q)}(P) = Q \tag{1}$$

To see this, notice that, by (I3), the plane $\mu(P, Q)$ contains three non-collinear points, say P_1, P_2, P_3 , and $PP_i \equiv QP_i$, for $i = 1, 2, 3$ by the definition of $\mu(P, Q)$, so by (C4) $s_{\pi(P_1, P_2, P_3)}(P) = Q$. Since, by (I4) $\pi(P_1, P_2, P_3) = \mu(P, Q)$ we have proved (1).

Let $ABCDE$ be a regular pentagon, with sides AB, BC, CD, DE, EA . Consider three consecutive vertices, say A, B, C .

If A, B, C are collinear, then, by (C8), B, C, D must be collinear as well. Let π be a plane that contains B, C, E (it exists by (I4)). Then A and D are in π as well (by (I2)), so the pentagon is planar.

If A, B, C are not collinear, then let π be the plane, which exists by (I4), containing these three points. Suppose $D \notin \pi$. Let $Q = s_{\mu(A, C)}(D)$ and $Q' = s_\pi(Q)$. Since $B \in \mu(A, C)$, we get from (C3) that $DB \equiv QB$, and since $DB \equiv EB$ by hypothesis, we get,

by (C1), $EB \equiv QB$. By (C5) and (1) we have $DA \equiv QC$ and $DC \equiv QA$, and since, by hypothesis $DA \equiv EC$ and $DC \equiv EA$, we get, by (C1) $EC \equiv QC$ and $EA \equiv QA$. Since we also have $EB \equiv QB$, and A, B, C are three non-collinear points, we get, by (C4) that $E = Q$ or $E = Q'$. Since neither Q nor Q' can be in π , by (C3) and (C6). \square

Theorem 1. *PRP follows from* $\{(I1)–(I4), (C1)–(C8), (B1)–(B3)\}$.

Proof. Suppose now that the regular pentagon $ABCDE$ is not planar. According to the above lemma, no four of its vertices can be coplanar, so using (B2) we deduce that there are three vertices of the pentagon, such that the plane π in which they lie is between one of the remaining vertices and the reflection of the other one in π .

Suppose that the three segments formed by the three vertices are two sides of the pentagon and a diagonal. We may assume w. l. o. g. that they are A, B, C , and that the other two vertices E, D given by (B2) are such that $s_\pi(E)\pi D$. Let $D' = s_{\mu(A,C)}(D)$. Then, by (B3) $s_\pi(E)\pi D'$. Since $B \in \mu(A, C)$, $DB \equiv D'B$ (by (C2) and (C3)), and since $DB \equiv EB$ by hypothesis, we have $EB \equiv D'B$ (by (C1)). We also have $EC \equiv D'C$ (since $DA \equiv D'C$ (by (C5) and (1)) and $EC \equiv DA$ by hypothesis) and $EA \equiv D'A$ (since $DC \equiv D'A$ (by (C5) and (1)) and $EA \equiv DC$ by hypothesis), so $E = D'$ or $E = s_\pi(D')$ (by (C4), which may be applied since A, B, C cannot be collinear, else four vertices of the pentagon would be coplanar). However, the latter cannot hold, for then $s_\pi(E)$ would be D' (by (C3), s_π is an involution) and we would have $D'\pi D'$, contradicting (B1). Thus $E = D'$ and, by (C7) and (1), A, C, D, E are coplanar, a contradiction.

We are left with the second possibility, namely that three segments formed by the three vertices given by (B2) are two diagonals and a side of the pentagon. W. l. o. g. we may assume that they are A, B, D and that the other two vertices E and C given by (B2) are such that $s_\pi(E)\pi C$, π being $\pi(A, B, D)$. Let $C' = s_{\mu(A,B)}(C)$. Then, by (B3), $s_\pi(E)\pi C'$. Since $D \in \mu(A, B)$, $CD \equiv C'D$ (by (C2) and (C3)), and since $CD \equiv ED$ by hypothesis, we have $ED \equiv C'D$ (by (C1)). We also have $EA \equiv C'A$ (since $CB \equiv C'A$ (by (C5) and (1)) and $CB \equiv EA$ by hypothesis) and $EB \equiv C'B$ (since $CA \equiv C'B$ (by (C5) and (1)) and $EB \equiv CA$ by hypothesis), so $E = C'$ or $E = s_\pi(C')$ (by (C4), which may be applied since A, B, D cannot be collinear, else four vertices of the pentagon would be coplanar). However, the latter cannot hold, for then $s_\pi(E)$ would be C' , and we would have $C'\pi C'$, contradicting (B1). Thus $E = C'$ and, by (C7) and (1), A, B, C, E are coplanar, a contradiction. \square

Notice that we have not asked that the collinearity relation satisfies the usual axioms (such as $\Lambda(A, B, C) \wedge \Lambda(A, B, D) \wedge C \neq D \rightarrow \Lambda(A, C, D)$), nor did we ask the betweenness notion Σ to satisfy even the rudimentary requirements of [30], though we have asked it to satisfy (B2) and (B3), which are quite special. There is also no mention of 3-dimensionality in the incidence axioms, in particular we are not requiring the existence of four non-coplanar points, nor do we stipulate that if two planes have a point in common,

then they have a line. The two axioms that ensure that the “dimension” is 3 are (I4) and (C2). There is, of course, no reasonable dimension notion which can be defined based on {(I1)–(I4), (C1)–(C8), (B1)–(B3)}, but if we think of what the dimension of any reasonable geometry based on an extension of the above axiom system would be, we get that “planes” would have to be, by (I4), 2-dimensional subspaces, and by (C2) that hyperplanes of that geometry are “planes”, so the dimension must be 3. However, a model of our axiom system — which could be called neither a “reasonable geometry” nor “3-dimensional” — consists of all the point in the real plane, with the real plane as the only “plane” individual, with usual point-plane incidence (all points are incident with the one and only plane) and collinearity, but with a congruence relation under which $AB \equiv CD$ for all points A, B, C, D , and with vacuous Σ .

It is plain that these axioms are satisfied by a variety of three-dimensional ordered spaces, such as ordered regular Fanoian affine-metric spaces, absolute spaces, in the sense defined in Konrad [13, I.5, p. 40–48] (i. e. all planes are Hilbert planes, i. e. they satisfy all plane axioms of groups I,II, III of Hilbert’s *Grundlagen der Geometrie*, and the space is an ordered three-dimensional space), generalized hyperbolic spaces over arbitrary ordered fields (models of which can be represented as the interior of a sphere in three-dimensional Euclidean space over the ordered field F , with congruence defined as usual, e. g. in [36, p. 48f], $AB \equiv CD \leftrightarrow \Psi(\mathbf{a}, \mathbf{b}) = \Psi(\mathbf{c}, \mathbf{d})$, with \mathbf{x} representing X , and $\Psi(\mathbf{x}, \mathbf{y}) = (1 - \mathbf{x} \cdot \mathbf{x}) \cdot (1 - \mathbf{y} \cdot \mathbf{y}) \cdot (1 - \mathbf{x} \cdot \mathbf{y})^{-2}$, $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$, $\mathbf{x} = (x_1, x_2, x_3)$), and double elliptic spaces (i. e. the natural geometry of the sphere \mathbb{S}^3 , cf. [18, Ch. VII]).

Notice also that the PRP remains true even in geometries whose models are embeddable in models of the axiom system {(I1)–(I4), (C1)–(C8), (B1)–(B3)}, since PRP is a universal sentence, and thus is preserved under submodels.

This proof of PRP is not entirely satisfactory, as it uses the notion Σ and the axioms (B1)–(B3) to prove a statement which is formulated entirely in terms of segment congruence and incidence. One would expect to see a proof that uses only the notions \in , \equiv , and Λ . This expectation was explicitly stated by Hilbert [9, p. 27]:

In der modernen Mathematik (wird) solche Kritik sehr häufig geübt, woher das Bestreben ist, *die Reinheit der Methode zu wahren, d. h. beim Beweise eines Satzes womöglich nur solche Hilfsmittel zu benutzen, die durch den Inhalt des Satzes nahe gelegt sind.*

We now turn our attention to such a methodologically pure proof of PRP.

3 The Second Proof

The language in which the axiom system will be expressed is a sublanguage of that of the first proof obtained by removing the notion Σ .

The first group of axioms are the incidence axioms (I1), (I2), (I4), and:
(I1’) If A, B, C and A, B, D are collinear, and $C \neq D$, then A, C, D are collinear as well.
(This axiom allows us to say that point C is on *line* \overline{AB} , whenever $\Lambda(A, B, C)$)

The second group of axioms are the congruence axioms (C1), (C2) and:

(C3') every segment has a midpoint, which is a point, collinear with and equidistant from the endpoints of the segment.

We write $m(AMB)$ whenever M is a midpoint of AB . Notice that we have not postulated the uniqueness of the midpoint and cannot prove it inside this axiom system.

(C4') The medians corresponding to congruent sides of two congruent (all sides pairwise congruent) triangles are congruent.

(C5') If A, B, C, D, E are 5 distinct points and $m(AMD), m(BNC), m(EM'C), m(AN'B), \Lambda(M, N, E), \Lambda(M', N', D)$, then there is a point P which is both on line \overline{MN} and on line $\overline{M'N'}$.

This axiom holds in any geometry in which a material system of 5 points with equal masses has a centre of mass, since that centre of mass of A, B, C, D, E (with equal masses attached to each) will have to be P .

(C7') If the perpendicular bisector planes of AB and CD coincide, then A, B, C, D lie in the same plane.

(C8') If B is a midpoint of AC , and $AC \equiv A'C', BA \equiv B'A', BC \equiv B'C'$, then B' is a midpoint of $A'C'$.

Theorem 2. *PRP follows from $\{(I1), (I1'), (I2), (I4), (C1), (C2), (C3'), (C4'), (C5'), (C7'), (C8')\}$.*

Proof. The proof is based on the proof of B. L. van der Waerden [38].

Let $ABCDE$ be the pentagon, with AB, BC, CD, DE, EA for sides, and suppose its vertices are not all in one plane. If any four of them did lie in a plane, then A, B, C, D would be in π_1 and A, B, C, E in π_2 , A, C, D, E in π_3 , A, B, D, E in π_4 , and B, C, D, E in π_5 . If any three points, say A, B, C , were not collinear, then from the first two relations we would deduce that D and E are in $\pi(A, B, C)$, and the pentagon would be planar. The same conclusion would be reached, using (I1'), (I2) and (I4), if all triples of points were collinear.

Thus there are four vertices, say A, B, C, D , which do not lie in one plane.

Let M and N be midpoints of AD and BC (C3'). Using the fact that the pentagon is equilateral and isogonal, as well as (C2) and (C4'), we deduce that the points E, M, N lie on both $\mu(A, D)$ and $\mu(B, C)$, and since A, B, C, D are not coplanar, these two planes cannot coincide (by (C7')). Thus, if they are different, the points E, M, N must be collinear (by (I4)). They are indeed different, for if M were equal to N , then A, B, C, D would be coplanar (by (I4) and (I2)), and E cannot be either M or N , as (C8') together with (I2) would imply, in case $E = M$ or $E = N$, that all vertices of the pentagon are collinear, and thus coplanar, (by (I2) and (I4)).

Notice that C, B, A, E cannot be coplanar, for if they were, the line EN would have to lie in it, and thus M as well, and that would force D to lie in the same plane, since

M is the midpoint of AD (by (C3') and (I2)). We can now repeat the same argument we have used for A, B, C, D and E to E, A, B, C and D , to obtain that D, M', N' are collinear, where M' and N' are midpoints of EC and AB . By (C5') the lines \overline{MN} and $\overline{M'N'}$ intersect. The incidence axioms now imply that E, D, M, N, M', N' lie in the same plane. Using (I2) and (C3'), we deduce that, since M is a midpoint of AD , A also belongs to this plane, and since N' is a midpoint of AB , B also belongs to this plane, and since N is a midpoint of BC , C must also belong to this plane, thus A, B, C, D, E are coplanar, which contradicts our initial assumption. □

This proof differs from both van der Waerden's [38] and Coxeter's and Bol's [39] proofs, in that it does not assume the existence of a centre of mass and of a group of isometries which interchanges vertices of the given pentagon, but only (C5'), which is a significant weakening of the requirement that a centre of mass exists for all discrete material systems.

As we mentioned earlier, for PRP to be true in a particular geometry, it is enough for the models of that geometry to be embeddable in a geometry satisfying our axioms.

Thus, this proof remains valid in any 3-dimensional metric geometry whose incidence structure is embeddable in an affine space and whose field is of characteristic different from 2 and 5 (in the case of Euclidean metric spaces, the restriction $\neq 2$ on the characteristic is not needed, as all 3-dimensional Euclidean spaces are Fanoian, as shown in [15]; we ask the characteristic to be different from 5, to ensure the existence of a center of mass for a quintuple of points). Such are (3-dimensionality and the restriction on the characteristic is valid in all cases): regular Fanoian affine metric spaces, which were provided with a very simple axiomatisation in [26] (see also [27], [28], and for a particularly simple axiom system of 3-dimensional Euclidean spaces, [25]), Stanik's [33] rectangular spaces, as well the ordered generalized hyperbolic spaces encountered earlier, and the double elliptic spaces.

To deduce from our second proof that PRP holds in the last two types of spaces, notice that Galperin [17] (see also Wiegand [40], [41]) has shown the existence of a centre of mass for discrete material systems in both n -dimensional hyperbolic geometry over the real number field and in the double elliptic geometry on \mathbb{S}^n (the existence of the mass centre had been shown earlier, by Perron [24], in the case of 3-dimensional hyperbolic geometry over the reals). By Tarski's theorem, which states that any elementary sentence that holds in the field of real numbers must hold in all real closed fields, the pentagon theorem remains valid in the 3-dimensional hyperbolic geometry and in the double elliptic one over any real-closed field, and thus it remains valid in generalized 3-dimensional hyperbolic geometry and in double elliptic geometry over any ordered field F , since any model of these can be extended to a model of the same geometry over the real closure of F , where PRP is true, and thus it must hold in the submodels as well, since it is a universal sentence, and as such preserved under submodels.

4 Conclusion

As one can see the two sets of axioms are incomparable. One cannot say that one is weaker than the other, and it may be that interesting geometries are subgeometries of geometries satisfying the axioms of the first system, but not of the second. So far, a larger class of known geometries could be proved to satisfy PRP by means of the second proof than by means of the first proof, for it seems that most known geometries are embeddable in one satisfying the axioms $\{(I1), (I1'), (I2), (I4), (C1), (C2), (C3'), (C4'), (C5'), (C7'), (C8')\}$.

There are, however, many geometries in which we could not prove that PRP holds. An example is the 3-dimensional hyperbolic geometry in which every plane is a generalized hyperbolic plane in the sense of Klingenberg [12], and the space is an ordered three-dimensional space, a geometry introduced in [16, §3,4].

It would also be of considerable interest to know whether PRP is true in much more general three-dimensional geometries, such as those axiomatized in [1] and [20] (or the more general ones from [6] and [7]), provided that the coordinate field is subject to some constraints on the characteristic. We know that PRP holds in all geometries axiomatized in [1] and [20] whose projective extensions satisfy $(C5')$, but we have no reasonable description of those spaces.

It would also be nice to have some metric geometry in which PRP does *not* hold.

These questions might be settled by looking at the assumptions underlying a third proof of PRP, given by Korchmáros [14], for that proof uses up to its last lines only facts which hold in all reflection geometries, such as [1] and [20], except for one final assumption, which allows one to conclude that if ϱ , the composition of two symmetries (in planes or lines), is such that $\varrho(A) = C$, $\varrho(C) = E$, $\varrho(E) = B$, $\varrho(B) = D$, $\varrho(D) = A$, then A, B, C, D, E are coplanar. It is not clear what the minimal geometric assumptions must be for this statement to hold, nor whether those would have to include $(C5')$.

We know that PRP does not hold in 3-dimensional Minkowski geometry (the geometry of normed vector spaces, see [37]), for the pentagon with vertices $(0, -1, 1), (1, 0, 0), (0, 1, -1), (-1, 0, -1), (-1, -1, 0)$ is regular with the maximum norm $\|(x, y, z)\| = \max\{|x|, |y|, |z|\}$, but not planar.

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