Contents

Chapter 1

PRELIMINARY RESULTS

1.1 History

> L'invention de la géométrie a consisté à introduire le langage
> et les opérations logiques du langage comme developpement
> d'une connaissance intuitive. Ce pas est énorme.
>
> Paul Valéry, *Cahiers*.

Most of the theorems of Euclidean geometry were written down by EUCLID in Books I-VI, XI-XIII of his *Elements*, but although the *axioms* of Euclidean geometry are etymologically rooted in the $\alpha\xi\iota\omega\mu\alpha\tau\alpha$ (a term used by ARISTOTLE, for which EUCLID uses $\kappa o\iota\nu\alpha\iota$ $\varepsilon\nu\nu o\iota\alpha\iota$ (common notions)), listed together with the $\alpha\iota\tau\eta\mu\alpha\tau\alpha$ (postulates) in Book I of the *Elements*, their contemporary understanding is fundamentally different from that of 'the Greeks'. EUCLID did not develop geometry axiomatically, nor did he intend to do that, if we give axioms the contemporary meaning (cf. [74], [81], [26]).

The first gapless axiomatic development of Euclidean geometry was achieved by D. HILBERT [35]. By aiming at doing justice to the tradition of EUCLID's geometric terminology, HILBERT axiomatized geometry in a three-sorted language, with variables to be interpreted as 'points', 'lines' and 'planes', with predicates for 'incidence' (a binary predicate between points and lines, as well as between lines and planes), 'betweenness' (a ternary predicate between points), 'segment congruence' (a quaternary predicate between points) as well as 'angle congruence' (a senary predicate between points). Ever since it appeared in 1899 there have been numerous efforts aimed at simplifying HILBERT's axiom system. These simplifications have amounted to either proving that a particular axiom is superfluous, or to showing that only a special case of a certain axiom is really needed or else to providing a completely different axiomatization in a different language, possibly with different intended interpretations of the individual variables.

Among the latter, most notable for the simplicity of the language and of the axioms was the axiom system proposed by A. TARSKI, first published in 1948, then in 1959 ([90]) and then in final version in 1983 ([72]). TARSKI's axiom system is expressed in a one-sorted first-order language[1] $L_{B\equiv}$, with individuals to be interpreted as 'points' and with two predicates,

[1] We shall write $L_{\cdots}$ for the first-order language with symbols $\cdots$.

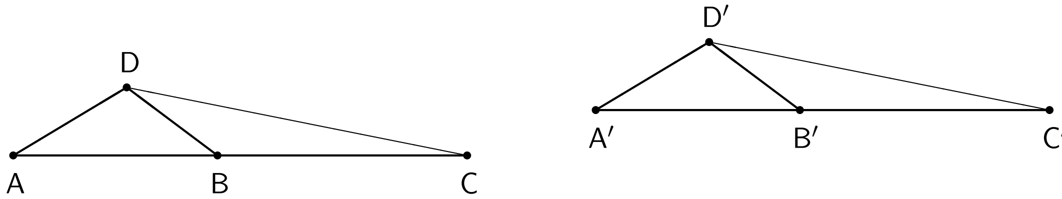

Figure 1.1: The five segment axiom (A1.1.5)

a ternary one, B (with $B(abc)$ to be read as 'point b lies between a and c') and a quaternary one, $\equiv$ (with $ab \equiv cd$ to be read as 'a is as distant from b as c is from d', or equivalently 'segment ab is congruent to segment cd')[2]. We shall use the following abbreviation for the concept of collinearity

$$L(abc) \overset{\text{def}}{\leftrightarrow} B(abc) \vee B(bca) \vee B(cab). \tag{1.1}$$

The axioms, whose statements will be explained immediately afterward, are:[3]

A 1.1.1 $ab \equiv ba,$

A 1.1.2 $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs,$

A 1.1.3 $ab \equiv cc \rightarrow a = b,$

A 1.1.4 $(\forall abcq)(\exists x)\,[B(qax) \wedge ax \equiv bc],$

A 1.1.5 $a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \rightarrow cd \equiv c'd',$

A 1.1.6 $B(aba) \rightarrow a = b,$

A 1.1.7 $(\forall abcpq)(\exists x)\,[B(apc) \wedge B(bqc) \rightarrow B(pxb) \wedge B(qxa)],$

A 1.1.8 $(\exists abc)\,\neg L(abc),$

A 1.1.9 $p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \rightarrow L(abc),$

A 1.1.10 $(\forall abcdt)(\exists xy)\,[B(adt) \wedge B(bdc) \wedge a \neq d \rightarrow B(abx) \wedge B(acy) \wedge B(xty)],$

A 1.1.11 [Continuity axiom schema] $(\exists a)(\forall xy)\,[\alpha(x) \wedge \beta(y) \rightarrow B(axy)]$
$\rightarrow (\exists b)(\forall xy)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)],$

where $\alpha(x)$, $\beta(y)$ *are formulas in* $\mathrm{L}_{B\equiv}$ *with* a, b, y *not occurring free in* $\alpha(x)$ *and* a, b, x *not occurring free in* $\beta(y)$.

A1.1.4 is a segment transport axiom, stating that we can transport any segment on any given line from any given point; A1.1.5 is the 'five-segment axiom', whose statement is close

[2] In [90] Tarski had used the notation $\delta(abcd)$ instead of $ab \equiv cd$ and $\beta(abc)$ instead of $B(abc)$, but switched to the one used in this thesis in [72].

[3] We shall omit to write the universal quantifiers in the case of universal axioms.

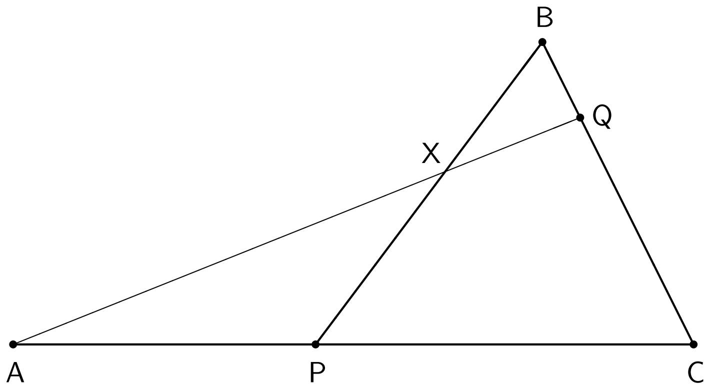

Figure 1.2: The PASCH axiom (inner form) (A1.1.7)

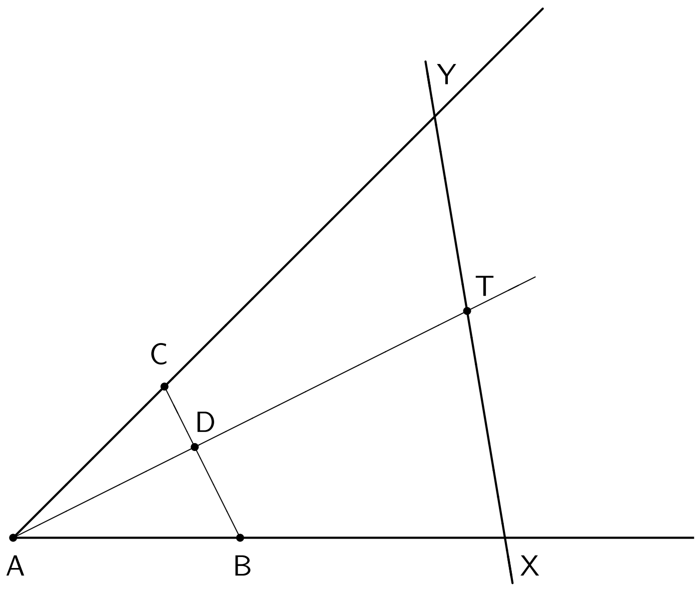

Figure 1.3: Through any point inside an angle, there is a line that meets both of its sides (A1.1.10)

to the statement of the side-angle-side congruence theorem for triangles; A1.1.7 is the PASCH axiom (in its 'inner form', i. e. if a line intersects a side of a triangle and the extension of another side, such that the intersection points lie on different sides of the third side of the triangle, then it must intersect that third side as well, the intersection point being between the first two intersection points); A1.1.8 is a 'lower dimension axiom' stating that the dimension is ≥ 2; A1.1.9 is an upper-dimension axiom, stating that the dimension is ≤ 2; and A1.1.10 is the Euclidean parallel axiom (in a variant that was first stated by J. F. LORENZ in 1791). It states, assuming the other axioms, in particular the order axioms, that *Through every point in the interior of an angle, there is a line intersecting the sides of that angle.*

TARSKI [72] proves the following representation theorem for $\mathcal{E}_2'' = Cn(\text{A1.1.1} - \text{A1.1.11})$.[4]

Representation Theorem 1.1.1 *$\mathfrak{M} \in Mod(\mathcal{E}_2'')$ iff $\mathfrak{M} \simeq \mathfrak{C}_2(F)$, for some real closed field F, where $\mathfrak{C}_2(F) = \langle F \times F, \mathbf{B}_F, \equiv_F \rangle$,*
$B_{\mathfrak{M}}(\mathbf{abc})$ iff $(\exists t \in F)\ 0 \leq t \leq 1, \mathbf{b} - \mathbf{a} = t(\mathbf{c} - \mathbf{a})$,
$\mathbf{ab} \equiv_F \mathbf{cd}$ iff $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|$,
with $\|(x_1, x_2)\| = x_1^2 + x_2^2$, for $x_1,\ x_2 \in F$.[5]

In [90] he singles out two fragments of $\mathcal{E}_2''$ that are interesting both historically and from the point of view of geometric constructions with various elementary instruments. They are $\mathcal{E}_2 = Cn(\text{A1.1.1 - A1.1.10})$ and $\mathcal{E}_2' = Cn(\text{A1.1.1 - A1.1.10, A1.1.12})$, where

A 1.1.12 $(\forall abcpqr)(\exists x)\,[B(cqp) \wedge B(cpr) \wedge ca \equiv cq \wedge cb \equiv cr \rightarrow cx \equiv cp \wedge B(axb)]$.

$\mathcal{E}_2$ is the Euclidean geometry that one gets by trying to imitate EUCLID's 'axiomatics'. It is an ordered plane Euclidean geometry with free mobility (i. e. it satisfies A1.1.4); it is the geometry of ruler and gauge constructions. The theorems proved by EUCLID in the *Elements* are however not all in $\mathcal{E}_2$, as he frequently assumes that lines that go through some inner point of a circle intersect that circle (which is in essence the statement A1.1.12 makes). All the theorems proved by EUCLID are in $\mathcal{E}_2'$, which is the geometry of ruler and compass constructions. TARSKI [90] also proves the following representation theorems for $\mathcal{E}_2$ and $\mathcal{E}_2'$.

Representation Theorem 1.1.2 *$\mathfrak{M} \in Mod(\mathcal{E}_2)$ iff $\mathfrak{M} \simeq \mathfrak{C}_2(F)$, for some Pythagorean ordered field F (i. e. an ordered field for which $F^2 + F^2 = F^2$).*
$\mathfrak{M} \in Mod(\mathcal{E}_2')$ iff $\mathfrak{M} \simeq \mathfrak{C}_2(\mathfrak{F})$, for some Euclidean ordered field F (i. e. an ordered field for which $F_{\geq 0} = F^2$).

Since there are Pythagorean ordered fields that are not Euclidean (a fact that seems to have been first noted in [35] (cf. also [11])), and Euclidean fields that are not real closed, we have $\mathcal{E}_2 \subset \mathcal{E}_2' \subset \mathcal{E}_2''$.[6]

$\mathcal{E}_2''$ is complete, decidable and not finitely axiomatizable (cf. [90]), whereas both $\mathcal{E}_2$ and $\mathcal{E}_2'$ are undecidable (cf. [93]). Both primitives B and $\equiv$ are needed for $\mathcal{E}_2$ (because there are Pythagorean fields that can be ordered in more than one way), but B is definable in terms

[4] $Cn(\Sigma)$ shall stand throughout the thesis for 'the set of logical consequences of a set of axioms Σ' and $Mod(\mathcal{T})$ for the class of all models of $\mathcal{T}$.

[5] $\mathfrak{C}_2(F)$, which may be defined over any ordered field F, will be called a *Cartesian plane over F*.

[6] Henceforth $\subset$ stands for 'strict inclusion'.

of $\equiv$ in $\mathcal{E}_2'$ (and hence in $\mathcal{E}_2''$), corresponding to the algebraic fact that Euclidean fields can be ordered in only one way, namely the positives have to be the squares.

Being unstable, $\mathcal{E}_2''$ has, for every infinite cardinal κ, 2^κ non-isomorphic models of cardinality κ (cf. [75], [76]).

1.2 Simplicity

We stated that TARSKI's axiomatization of Euclidean geometry stands out among the numerous simplifications of HILBERT's axiom system by the simplicity of the language and of its axioms.

The question of what exactly it is that makes it simple and whether it could be simplified further is an all too natural one, since these axiomatizations were proposed in order to offer a *simpler* axiom system than HILBERT's original one. If such a simplified axiom system is stated in the same language as HILBERT's original axiom system and is a subset of the latter, then we naturally call it simpler. Axiom systems with fewer axioms have been preferred already by ARISTOTLE ("Other things being equal, that proof is better which proceeds from fewer postulates or hypotheses or propositions." (*Anal. post.* I. 25, 86 a 33-35)) and WILLIAM OF OCKHAM's razor (*"Frustra fit per plura quod potest fieri per pauciora"*). But what if the axiom system has different axioms or is stated in a different language? And what if we want to know not whether an axiom system is *simpler* than another one, but *most simple*, or to put it differently, what if we want to have an axiom system simpler than *all* other possible axiom systems, one that is no longer simplifiable? In order to provide an answer to these questions we need an absolute criterion of simplicity. It is reasonable to ask for such an absolute criterion if simplifications of existing axiom systems are to be taken seriously, for otherwise there is no final destination one at least aims to reach by simplifying existing axiom systems. Without absolute criteria for simplicity one would not know when to stop simplifying.

Before turning to absolute simplicity criteria, we should like to emphasize that, since axiom systems axiomatize a structure that has been already constructed (at least partially, i. e. we already have some models of it), we consider that asking for 'natural' axioms (revealed by some infallible intuition, as required by ARISTOTLE (*Anal. post.* II. 29, 100 b 6), or otherwise), stems from the illusion that an axiom system brings the theory and its structures into being, an illusion that originates in the confusion between *Entstehungsgrund* and *Begründung*.

In our investigations, we consider an axiom system to be a tool for logically organizing an — at least partially — pre-existing theory (for which we have already constructed a model), a tool that comes discovery-wise last, although deduction-wise first. Or, as N. HARTMANN [32] put it

> Denn es liegt im Wesen der Fundamente, daß sie nur im Rückschauen von dem aus, was auf ihnen beruht, sichtbar werden können.

For A. TARSKI [89, p. 98] the choice of the language and of the axioms is also largely arbitrary:

> Ce ne sont pas d'ordinaire des considérations d'ordre théorique, fondamental, qui décident du choix d'un système déterminé de termes primitifs et d'axiomes

> parmi tous les systèmes équivalents: ce sont plutôt des raisons d'ordre pratique, didactique ou même esthétique.

We like to think of simplicity as of one such aesthetical reason for choosing a particular axiom system or a particular language.

Complete independence of an axiom system, i. e. each axiom is independent of all the others, is definitely a desirable property of any simple axiom system. Such axiom systems may, however, be further simplifiable, by replacing an axiom with a special instance of it, so in order to call an axiom system most simple, we need to ask for more than just complete independence. It is also desirable that the quantifier prefixes of the axioms are as simple as possible, that is there should be as few quantifier alternations as possible. Universal axioms are in this sense best possible, $\forall\exists$-axioms are next best, etc.

We now turn to some simplicity criteria, which will be relevant for the coming chapters (for other criteria cf. [49]).

Simplicity Criterion 1.2.1 *An axiom system Σ for a finitely axiomatizable theory $\mathcal{T}$ in a first-order language* L*, is* simple *and $\mathcal{T}$ has* simplicity degree $sd(\mathcal{T}) = n$*, if every axiom in Σ, when written in prenex form, has at most n variables, and there is no axiom system for $\mathcal{T}$ all of whose axioms, when written in prenex form, have at most $n-1$ variables.*

We shall provide in this thesis simple axiom systems for several geometries.

Let Σ be a finite axiom system for a theory $\mathcal{T}$ and $\alpha \in \Sigma$ an axiom. We call α *primary* if α cannot be split into two strictly weaker axioms β and γ that could replace it, except for splittings of the type $\beta = \alpha \vee \delta$, $\gamma = \alpha \vee \neg\delta$, for some sentence δ[7]i. e. if
for all sentences β and γ in the language of $\mathcal{T}$, if
$\Sigma \setminus \{\alpha\} \vdash \alpha \leftrightarrow \beta \wedge \gamma$ and
$\Sigma \setminus \{\alpha\}, \beta \nvdash \alpha, \quad \Sigma \setminus \{\alpha\}, \gamma \nvdash \alpha,$ then
$\Sigma \setminus \{\alpha\} \vdash \beta \vee \gamma.$[8]
The very fact that all interesting finite axiom systems have more than one axiom can be considered to be a measure of the interest mathematicians have in splitting axioms, for all those axiom systems could be expressed as a single axiom formed by the conjunction of all the axioms in that axiom system. One would probably find that tasteless because of the many 'and's the axiom contains, but even axioms that are not visibly a concatenation of different axioms *may in fact be it.* The disadvantage in having such concatenated axioms, that 'state too much', is that one would no longer be able to follow the step-by-step development (called *Stufenaufbau* in [57]) of that theory, and would no longer be able to know what the alternatives to some axioms would be (like non-Euclidean geometry, Pasch-free geometry, etc.). Two axioms, α and α' will be called *conjugate* iff $\alpha \wedge \alpha'$ is primary in $(\Sigma \setminus \{\alpha, \alpha'\}) \cup \{\alpha \wedge \alpha'\}$. O. Helmer [33] has proposed the following criterion of simplicity, with the aim of having axioms with "as little content as possible" ([33]):

[7] Splittings of this type will be called *trivial.*

[8] This last condition is equivalent to the one used by Helmer, that $\Sigma \setminus \{\alpha\} \vdash \beta \leftrightarrow \alpha \vee \delta, \quad \Sigma \vdash \gamma \leftrightarrow \alpha \vee \neg\delta$ for some sentence δ. An equivalent formulation of this definition of a 'primary' axiom would be: α is primary iff in the Lindenbaum - Tarski algebra of $\Sigma \setminus \{\alpha\}$, there are at most 4 elements that are $\leq \neg\alpha$, or equivalently iff $\Sigma \setminus \{\alpha\} \cup \{\neg\alpha\}$ has at most 2 completions.

Simplicity Criterion 1.2.2 *A finite axiom system* Σ *is* semantically simple *if all its axioms are primary and if it contains no pair of conjugate axioms.*

We are very far from having an axiom system for any fragment of Euclidean geometry that would be semantically simple. The reason we nevertheless stated this criterion, is that we consider it of fundamental importance, although were able to make only a small step on the road that would lead to a semantically simple axiom system for Euclidean geometry, by splitting the Euclidean parallel axiom.

The previous two criteria apply to axiom systems in a given first-order language. As we know, the 'same' theory can be axiomatized in different languages, even with different intended interpretations of the individual variables, which in turn may be many-sorted. We feel that the language itself should be an object of aesthetic interest, so we find that there is a need for a criterion for the simplicity of the language. This criterion should also be able to incorporate other requirements that we may impose upon the axiom system. From a constructive point of view, an important requirement would be to have only universal axioms in a language which contains only operation symbols (constant symbols may be considered to be 0-ary operation symbols). In the case of Euclidean geometry such an axiomatics would be very much in the spirit of 'Greek' geometry, for which the geometric construction (by ruler and compass) was the only means of proving the 'existence' of a certain point (cf. [92], [25], [74], [91], [80]). We shall however ignore, throughout this thesis, axiomatizations of Euclidean geometry in languages that have more than one sort of variables, or in languages where the individual variables have semantic interpretations different from 'points' (cf. [49] for a survey of such axiomatizations). In order to state such a simplicity criterion, we first have to specify what we mean by 'Two axiom systems in two different first-order languages axiomatize the 'same' theory', or to put it differently, when are two theories essentially equivalent. The usual answer would be: 'When the two theories are synonymous' (as defined in [16]), i. e., when the two theories have a common definitional extension. We find however, that this definition allows too many theories to pass for 'geometries'. The standard axiom system for the first-order theory of the field of complex numbers with conjugation, from which we discard the axiom schema that states that the field is algebraically closed, would, under such a definition, be an axiom system for the metric part of plane Euclidean geometry (i. e. for Euclidean geometry without order or free mobility). In order to avoid such anomalies, we also stipulate that the predicates and (non 0-ary) operations of the language of a theory to be called a 'Euclidean geometry' should be invariant under isometries[9] or under orientation-preserving isometries, if the geometry to be axiomatized is ordered with free mobility. The formal definitions are:

Definition 1.2.1 *Let* L *and* L' *be two one-sorted*[10] *first-order languages. Two theories* $\mathcal{T}$ *and* $\mathcal{T}'$*, in* L *and* L' *respectively, will be called* synonymous *if they have a common definitional extension* $\overline{\mathcal{T}}$ *in* $\mathrm{L}\cup\mathrm{L}'$*.*

Definition 1.2.2 *(i) A theory* $\mathcal{T}$ *in a first-order language* L *extending* $\mathrm{L}_{\equiv}$*, will be called* metric *if* $\equiv$ *satisfies A1.1.1, A1.1.2 and A1.1.3. (ii) A theory* $\mathcal{T}$ *in* $\mathrm{L}_{B\equiv}$ *will be called an* oriented plane geometry *if* $\mathcal{E}_2 \subseteq \mathcal{T}$*.*

[9] D. Scott [70] calls such predicates 'geometrical relations'.

[10] Throughout this thesis the language will be one-sorted, a fact we shall henceforth omit to mention.

Definition 1.2.3 *(i) A theory $\mathcal{T}$ in a first-order language* L *extending* $\mathrm{L}_{\equiv}$*, will be called* geometric *if its* $\mathrm{L}_{\equiv}$ *reduct* $\mathcal{T} \cap \mathrm{L}_{\equiv}$ *is metric and if all of its predicates and (non* 0*-ary) operations are invariant under isometries (i. e. if P is an n-ary predicate in* L*, and* $\mathfrak{M} \in \mathfrak{Mod}(\mathcal{T})$*, then* $\mathbf{P}_{\mathfrak{M}}(\mathbf{x}_1 \cdots \mathbf{x}_n)$ *iff* $\mathbf{P}_{\mathfrak{M}}(\varphi(\mathbf{x}_1) \cdots \varphi(\mathbf{x}_n))$*, for all* $\varphi : u(\mathfrak{M}) \to \mathfrak{u}(\mathfrak{M})$ *that satisfy* $\varphi(\mathbf{x})\varphi(\mathbf{y}) \equiv_{\mathfrak{M}} \mathbf{xy}$*; analogously for operations.)*[11]
(ii) A theory $\mathcal{T}$ in in a first-order language L *extending* $\mathrm{L}_{B\equiv}$ *will be called* oriented geometric *if its* $\mathrm{L}_{B\equiv}$ *reduct* $\mathcal{T} \cap \mathrm{L}_{B\equiv}$ *is an oriented plane geometry and if all of its predicates and (non* 0*-ary) operations are invariant under orientation-preserving isometries.*

By using, in this definition, the group of isometries and the group of orientation-preserving isometries, and not the larger groups of transformations that preserve the $\equiv$ relation (i. e. maps $\varphi : u(\mathfrak{M}) \to \mathfrak{u}(\mathfrak{M})$ that satisfy $\mathbf{xy} \equiv_{\mathfrak{M}} \mathbf{uv}$ iff $\varphi(\mathbf{x})\varphi(\mathbf{y}) \equiv_{\mathfrak{M}} \varphi(\mathbf{u})\varphi(\mathbf{v})$), respectively the group of orientation-preserving similarities (transformations that preserve $\equiv$, B and the orientation), we allow axiomatizations of geometry endowed with a unit distance (see §2.6). If the intention is to axiomatize Euclidean geometry where it is not possible to define a predicate for 'unit distance', then the groups of transformations in the above definition should be changed accordingly.

Definition 1.2.4 *(i) Two theories $\mathcal{T}$ and $\mathcal{T}'$, both expressed in first-order languages,* L *and* L′*, will be called* equivalent *if they are synonymous, such that $\overline{\mathcal{T}}$, their common definitional extension in* L∪L′*, is a geometric theory. (ii) A theory $\mathcal{T}$ in* L *will be called* orientation-equivalent *to an oriented plane geometry $\mathcal{T}'$ if the two theories are synonymous, such that $\overline{\mathcal{T}}$, their common definitional extension in* $\mathrm{L} \cup \mathrm{L}_{B\equiv}$*, is an oriented-geometric theory.*

Note that two theories may be orientation-equivalent without being equivalent.

For a given theory $\mathcal{T}$ in a first-order language L that is an extension of $\mathrm{L}_{\equiv}$, let $[\mathcal{T}]$ denote the set of all theories equivalent to $\mathcal{T}$. For a plane oriented geometry $\mathcal{T}$ let $[\mathcal{T}]'$ denote the set of all theories orientation-equivalent to $\mathcal{T}$. We are now ready to state our last simplicity criterion.

Simplicity Criterion 1.2.3 *Let $\mathcal{T}$ be a metric theory in a first-order language* L *that is an extension of* L_D*, or an oriented plane geometry. An axiom system Σ for a theory $\mathcal{G}$ in $[\mathcal{T}]$ (respectively in $[\mathcal{T}]'$), will be said to be* expressed in the simplest possible language*, if the arity a and number of the predicates and operations n it contains is minimal (i. e. $(a, n) \leq (a', n')$ for all pairs (a', n') of arities, numbers of predicates and operations, of theories in $[\mathcal{T}]$ (resp. in $[\mathcal{T}]'$), the ordering being lexicographic) . Σ is said to be* constructively expressed in the simplest possible language*, if Σ contains only universal axioms, is expressed in a language L_{con} with function symbols only, the arity of its operations is minimal, and if there is no constructive axiomatization in a strict sublanguage of L_{con} (or, equivalently if no operation is constructively definable from the others).*

[11]Operations will be required to be invariant under isometries unless they are defined as 'arbitrary'; this shortcoming could have been corrected, by assigning a certain point in a uniform way to all operations used. For example, for the circumcentre operation U, assigning the circumcentre of the $\triangle abc$ to the triple (a, b, c), whenever a, b, c are not collinear, and an arbitrary point, otherwise, we could have stipulated that $U(abc) = a$ whenever a, b, c are collinear. We chose not to do so in order to (1) emphasize the geometrical meaning of our operations, (2) avoid stating axioms like $L(abc) \to U(abc) = a$, that have no geometric content, (3) emphasize that we don't need decision operations, i. e. operations that enable us to decide whether the condition for the 'meaningful' definition of the operation is fulfilled or not.

It is known that one ternary relation (for example TARSKI's J, with $J(xyz)$ having the intended interpretation 'the distance betwenn x and y is less than the distance between y and z') can axiomatize any of $\mathcal{E}_2$, $\mathcal{E}_2'$ and $\mathcal{E}_2''$ (cf. [49]) and that binary relations can axiomatize in first-order logic not only none of the $\mathcal{E}$'s, but also none of the $\mathcal{UE}$'s[12] (cf. [64]), so we know what the simplest possible language is for all of the $\mathcal{E}$'s is.

The first constructive axiom system for a theory in $[\mathcal{E}_2]$ was provided in [46]. It was expressed in a language containing two quaternary operations and three individual constants. Some minor inconsistencies in [46] were corrected in [73], where we also find a constructive axiom system for a theory in $[\mathcal{E}_2']$ in a language with three quaternary operations and three individual constants. The question of the minimal arity of the operations with which several geometries may be axiomatized has been studied in [51], [52], [53], [54], and will be analyzed in detail in chapter 3.

Finally we can combine the Simplicity Criteria 1.2.1 and 1.2.3 to

Simplicity Criterion 1.2.4 *Let $\mathcal{T}$ be as in Simplicity Criterion 1.2.3. An axiom system Σ for a finitely axiomatizable theory in $[\mathcal{T}]$ (or in $[\mathcal{T}]'$), is* simple, regardless of language*, and $[\mathcal{T}]$ (resp. $[\mathcal{T}]'$) has* Simplicity degree *n, $Sd([\mathcal{T}]) = n$ (resp. $Sd([\mathcal{T}]') = n$), if every axiom in Σ, when written in prenex form, has at most n variables, and there is no axiom system for any theory in $[\mathcal{T}]$ (resp. $[\mathcal{T}]'$), all of whose axioms, when written in prenex form, have at most $n-1$ variables.*[13]

This last criterion asks, geometrically speaking, What is the minimum number of 'points' some axiom of a finite axiom system for Euclidean geometry, in an unspecified language, has to talk about?

We shall provide the answer to this question in the following chapters.

[12] See §2.6 for a definition of $\mathcal{UE}_2$.

[13] The language in which a theory in $[\mathcal{T}]$ is expressed should not contain individual constants. This restriction would eliminate any theory axiomatized by universal axioms, so one should allow for one purely existential sentence as an axiom to replace the use of constants. Without this restriction the theorems on the absolute simplicity of axiom systems for geometry in chapter 3 are no longer valid, for Scott's theorem on the minimal number of variables in an axiom fixing the dimension is no longer true if one has individual constants in the language. The sentence

$$(\forall x)(\exists yz)\,[\neg L(a_0a_1a_2) \wedge L(a_0a_1y) \wedge L(a_0a_2z) \wedge L(xyz)],$$

although it contains only three variables, is true in the Cartesian plane over $\mathbb{R}$, but not in any higher-dimensional space.

Chapter 2

THE CLASSICAL SETTING

2.1 An independent axiom system

Axiomatizations of geometry in languages with predicates will be called *classical*, since their axioms will require the use of the existential quantifier, which allows for a certain degree of 'non-constructiveness' in those axioms which are allowed to make 'existence statements'.

TARSKI's axiom systems for $\mathcal{E}_2$, $\mathcal{E}_2'$, that were presented in §1.1, are not known to be independent. Moreover, since most axioms are statements about both B and $\equiv$, we cannot get an axiom system for any interesting purely metric theory (i. e. one stated in terms of $\equiv$ only) by putting together all the $\mathrm{L}_{\equiv}$-axioms it contains. Since we shall be investigating in chapters 3 and 4 several purely metric theories, we find it appropriate to present an independent axiom system for $\mathcal{E}_2'$, that contains subsystems for plane Euclidean geometry over 2-formally real, ordered, and Pythagorean ordered fields.

The axiom system is $\Theta = \{$A1.1.1, A1.1.9, A2.1.1 - A2.1.11$\}$, where

A 2.1.1 $ab \equiv cd \rightarrow cd \equiv ab$,

A 2.1.2 $ab \equiv cd \wedge cd \equiv ef \rightarrow ab \equiv ef$,

A 2.1.3 $(\forall abca'b'c'xm)(\exists x')(\forall y)\,[a \neq b \wedge c \neq m \wedge ac \equiv am \wedge bc \equiv bm \wedge a'b' \equiv ab \wedge a'c' \equiv ac \wedge b'c' \equiv bc \rightarrow a'x' \equiv ax \wedge b'x' \equiv bx \wedge c'x' \equiv cx \wedge (a'y \equiv ax \wedge b'y \equiv bx \wedge c'y \equiv cx \rightarrow y = x')]$,

A 2.1.4 $(\forall abcd)(\exists m)\,[a \neq b \wedge c \neq d \wedge ac \equiv ad \wedge bc \equiv bd \rightarrow am \equiv cm \wedge bm \equiv cm \wedge m \neq a \wedge m \neq c]$

A 2.1.5 $(\exists abcde)\,[a \neq c \wedge b \neq d \wedge ab \equiv bc \wedge bc \equiv cd \wedge cd \equiv da \wedge ae \equiv be \wedge ce \equiv be \wedge ce \equiv de]$,

A 2.1.6 $(\forall abc)(\exists uv)\,[B(abc) \rightarrow u \neq v \wedge au \equiv av \wedge bu \equiv bv \wedge cu \equiv cv]$,

A 2.1.7 $B(abc) \rightarrow B(cba)$,

A 2.1.8 $B(abd) \wedge B(bcd) \rightarrow B(abc)$,

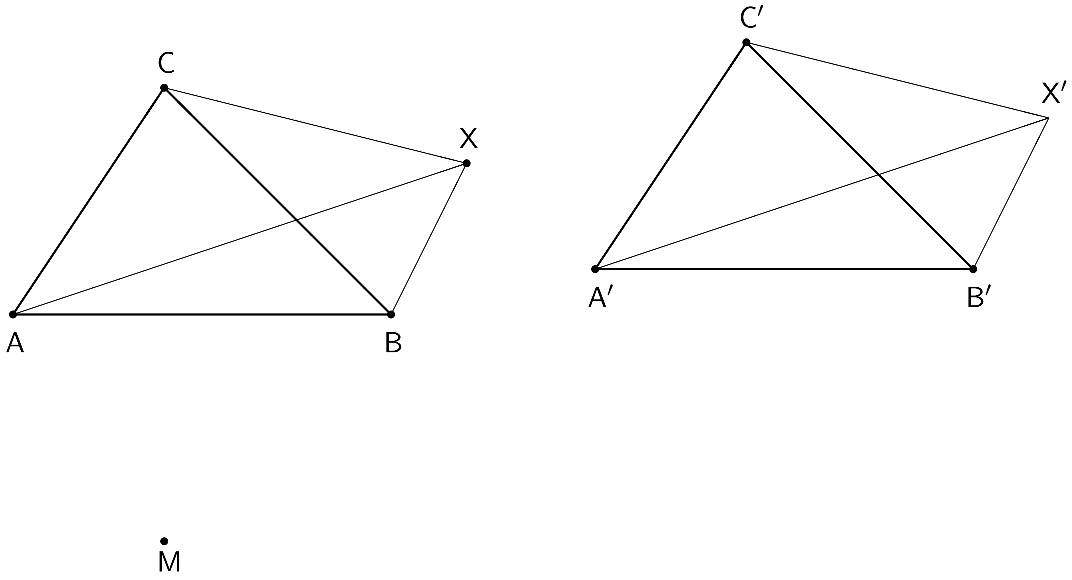

Figure 2.1: The rigidity axiom (A 2.1.3)

A 2.1.9 $(\forall abcdp)(\exists q)\,[B(apd) \wedge B(bdc) \rightarrow L(bpq) \wedge L(aqc)]$,

A 2.1.10 $(\exists abcd)(\forall m)(\exists s)\,[a \neq b \wedge ab \not\equiv cd \wedge (cm \equiv dm \rightarrow ab \equiv cs \wedge cm \equiv sm)]$,

A 2.1.11 $(\forall abcde)(\exists fghf'g'h')\,[d \neq a \wedge a \neq c \wedge b \neq a \wedge B(bac) \wedge B(eac) \wedge ae \equiv ac$
$\wedge\, de \equiv dc \rightarrow B(daf) \wedge B(daf') \wedge B(bfh) \wedge B(bf'h') \wedge fb \equiv fh \wedge f'b \equiv f'h'$
$\wedge\, gb \equiv gh \wedge g'b \equiv g'h' \wedge B(agc) \wedge B(ag'c) \wedge ag' \equiv gc]$.

A2.1.3 states that if two triangles $\triangle abc$ and $\triangle a'b'c'$ are congruent, then for any point x there is exactly one point x' whose distances from the vertices of $\triangle a'b'c'$ are congruent to the corresponding distances of x to the vertices of $\triangle abc$. In a transformational-geometric language this axiom will help in establishing that an isometry is uniquely determined by its values on three non-collinear points, i. e. given a map that maps three non-collinear points isometrically, one can extend that map (which was so far defined only for those three non-collinear points) uniquely to an isometry of the whole plane. There is also an upper-dimensional statement implicit in this axiom; its uniqueness statement says that the dimension is ≤ 2.

A2.1.4 states that the centre of the circumcircle exists for all triangles; it is a form of Euclid's parallel postulate; the requirement that the circumcentre should be different from the vertices of the triangle is needed as we do not have axiom A1.1.3 in our axiom system.

A2.1.5 states that there is a square and its diagonals intersect.

A2.1.9 is a weakened form of the Pasch axiom (the outer form, that says that if a line intersects a side of a triangle and the extension of another side, such that the two intersection points lie on the same side of the third side of the triangle, then it must intersect that third side as well, with information on where the intersection point lies), for instead of concluding $B(bpq) \wedge B(aqc)$ it just concludes $L(bpq) \wedge L(aqc)$, i. e. it just states that if a line intersects

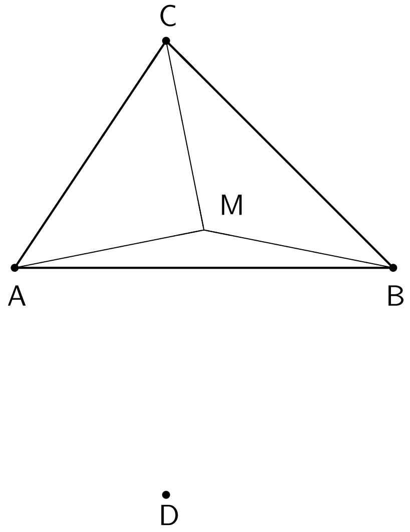

Figure 2.2: Every triangle has a circumcentre (A 2.1.4)

one side of a triangle and a certain extension of another, then that line cannot be parallel to the third side of the triangle.

To better understand the statement A2.1.10 makes, let

$$ab \leq cd \overset{\text{def}}{\leftrightarrow} (\forall m)(\exists s)\,[cm \equiv dm \to ab \equiv cs \wedge cm \equiv sm]$$

define the relation of inequality ($\leq$) between the lengths of segments.[1] If two segments ab and cd are congruent, then $ab \leq cd$ holds trivially (with $s = d$). A2.1.10 states that there are two non-degenerate segments, that are not congruent, such that the length of one of them is $\leq$ the length of the other. Its effect on the coordinate field is the same as that of a transport axiom, which, as an $L_\equiv$-sentence, will have to be expressed somewhat differently from A1.1.4 or A2.2.4, for example as (cf. [28, p. 606])

A 2.1.12 $(\forall apqc)(\exists b)\,[ap \equiv aq \wedge a \neq c \to bp \equiv bq \wedge ab \equiv ac]$.

It implies that the coordinate field is Pythagorean.

A2.1.11 is a complex geometrical statement, that is quite artificial, for it is geometrically phrased algebra: it states that the coordinate field satisfies

$$(\forall x)(\exists yz)\,[x \geq 0 \to x = y^2 + z^2].$$

Together with the requirement that the coordinate field should be Pythagorean, imposed by A2.1.10, this axiom is implying that the field is an Euclidean ordered field. If one is not interested in the step-by-step approach that was the raison d'être of A2.1.11, then one can

[1] Notice that the definition of $\leq$ is in terms of $\equiv$ only, as this is the only predicate used to axiomatize Euclidean geometry in [65].

just add an axiom stating that $ab \leq cd \vee cd \leq ab$ to $\mathcal{D}_2'$ (defined below) to get an axiom system for $\mathcal{E}_2' \cap \mathrm{L}_\equiv$ (cf. [65]).

Every axiom in Θ is independent of all the other axioms. The independence models, many of which are finite, can be found in [66] and [48]. Moreover, Θ contains subsystems for many interesting fragments of plane Euclidean geometry. With $\mathcal{BD}_2 \stackrel{\text{def}}{=} Cn(\Theta \setminus \{\text{A2.1.10}, \text{A2.1.11}\})$ and $\mathcal{D}_2' \stackrel{\text{def}}{=} Cn(\{\text{A2.1.1} - \text{A2.1.5}, \text{A1.1.1}\})$, we have the following

Representation Theorem 2.1.1
(i) $Cn(\Theta) = \mathcal{E}_2'$.
(ii) $Cn(\Theta \setminus \{\text{A2.1.11}\}) = \mathcal{E}_2$.
(iii) $\mathfrak{M} \in Mod(\mathcal{BD}_2)$ *iff* $\mathfrak{M} \simeq \mathfrak{C}_2(F)$*, for some ordered field* F.
(iv) $\mathfrak{M} \in Mod(\mathcal{D}_2')$ *iff* $\mathfrak{M} \simeq \mathfrak{D}_2(F)$*, for some 2-formally real field* F *(i. e.*
$x^2 + y^2 = 0 \rightarrow x = 0$*, or equivalently* $-1 \notin F^2$*), where* $\mathfrak{D}_2(F) = (F \times F, \equiv_F)$,
$\mathbf{ab} \equiv_F \mathbf{cd}$ *iff* $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|$,
with $\|(x_1, x_2)\| = x_1^2 + x_2^2$*, for* $x_1,\ x_2 \in F$.
(v) $Cn(\mathcal{D}_2', \text{A2.1.10}) = \mathcal{E}_2 \cap \mathrm{L}_\equiv$.

The most difficult part of the proof of this theorem, which is the proof of (iv), was carried out by R. SCHNABEL in [65]. The proof that all one needs in terms of order axioms is A1.1.9, A2.1.6, A2.1.7, A2.1.8, A2.1.9, was given by W. SZMIELEW in [87]. Putting the axioms together and providing the independence models was done by the author in [66], [48].

If, instead of asking that there is a square, one just asks that there should be a triangle (i. e. three non-collinear points) and a midpoint of a non-degenerate segment (say, of one of the sides of the triangle), formally

A 2.1.13 $(\exists abcc'm)(\forall m')\,[c \neq c' \wedge ac \equiv ac' \wedge bc \equiv bc' \wedge ma \equiv mb \wedge (ma \equiv m'a \wedge mb \equiv m'b \rightarrow m' = m)]$,

then we get, for the resulting theory $\mathcal{D}_2 \stackrel{\text{def}}{=} Cn(\{\text{A2.1.1 - A2.1.4, A1.1.1, A2.1.13}\})$, a non-standard formula for the Euclidean norm-function, as shown by

Representation Theorem 2.1.2 $\mathfrak{M} \in Mod(\mathcal{D}_2)$ *iff* $\mathfrak{M} \simeq \mathfrak{D}_2(F, k)$*, for some field* F *of characteristic* $\neq 2$*, and some* $k \in F$*, such that* $-k \notin F^2$*, where* $\mathfrak{D}_2(F, k)$*, which will be called a* Euclidean plane over (F, k)*, stands for* $\langle F \times F, \equiv_{(F,k)} \rangle$*, with*
$\mathbf{ab} \equiv_{(F,k)} \mathbf{cd}$ *iff* $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|$
with $\|(x_1, x_2)\| = x_1^2 + kx_2^2$*, for* $x_1,\ x_2 \in F$.

The constant k, which is determined only up to a quadratic factor (i. e. $\mathfrak{D}_2(F, k) \simeq \mathfrak{D}_2(F, ka^2)$ for all $a \in F \setminus \{0\}$) is called the 'orthogonality constant' (*Orthogonalitätskonstante* in [10, (§13,1)]), since the lines $ux+vy+w = 0$ and $u'x+v'y+w' = 0$ are orthogonal iff $kuu'+vv' = 0$. The proof of this theorem can also be found in [65]. A different axiom system for $\mathcal{D}_2$ can be found in [28] and axiom systems for synonymous geometric theories were proposed in [37] and [68]. One can also give an alternate description of the models of $\mathcal{D}_2$ in terms of Gaußian planes. Let L/K be a quadratic extension of a field K of characteristic $\neq 2$ and let $\{1, \sigma\}$ be its GALOIS group. The *Gaußian plane over* (L, K) is the structure $\mathfrak{G}(L, K) = \langle L, \equiv \rangle$, with $xy \equiv uv$ iff $\|x - y\| = \|u - v\|$, with $\|x\| = x\sigma(x)$, for $x,\ y,\ u,\ v \in L$. It generalizes the classical

Gaußian plane of complex numbers, $\mathfrak{M}(\mathbb{C},\mathbb{R})$, and we have $\mathfrak{G}(K(\sqrt{-k}),K) \simeq \mathfrak{D}_2(K,k)$, which allows us to use the 'Euclidean plane over (K,k)' and the 'Gaußian plane over (L,K)', with $L = K(\sqrt{-k})$, as synonyms.

2.2 The simple axiom system for $\mathcal{E}_2'$

Fünf ist
des Menschen Seele.
Wie der Mensch aus Gutem
und Bösem ist gemischt, so ist die Fünfte
die erste Zahl aus Grad' und Ungerade.
Friedrich Schiller, *Die Piccolomini.*

We shall present in this paragraph an axiom system Υ for $\mathcal{E}_2'$, all of whose axioms are statements about at most 5 points. In order to show that Υ is simple we need to show first that it is an axiom system for $\mathcal{E}_2'$, and then that there is no axiom system for $\mathcal{E}_2'$, all of whose axioms contain, in prenex form, at most 4 variables. It is worth noting that the axioms in Υ are not only simple from this syntactic point of view, but they were also chosen to be semantically weaker (individually) than traditional axioms for Euclidean geometry. We could have, for example, chosen $ab \equiv cd \rightarrow cd \equiv ab$ among the axioms, which would have eased the task of proving that the axiom system axiomatizes $\mathcal{E}_2'$, but we chose not to do so for the sake of having axioms that are individually weak. We also note that Υ can be easily made to consist of $\forall\exists$-sentences with at most 5 variables, by splitting A2.1.2 into two axioms, one stating the existence part and the other stating the uniqueness part of it. Let $\Upsilon = \{$A1.1.1, A1.1.8, A1.1.9, A2.1.7, A2.1.8, A2.2.1-A2.2.10$\}$ be an axiom system in $\mathrm{L}_{B\equiv}$, where (L is here the same abbreviation as in §1.1):

A 2.2.1 $a \neq b \wedge ((B(abc) \wedge B(abd)) \vee (B(abc) \wedge B(dab)) \vee (B(bca) \wedge B(bda))) \rightarrow L(acd),$

A 2.2.2 $ab \equiv cc \vee cc \equiv ab \rightarrow a = b,$

A 2.2.3 (i) $ab \equiv cd \wedge cd \equiv ce \rightarrow ab \equiv ce,$
(ii) $ab \equiv ac \wedge ac \equiv de \rightarrow ab \equiv de,$
(iii) $ab \equiv cd \wedge cd \equiv ae \rightarrow ab \equiv ae,$

A 2.2.4 $(\forall abc)(\exists d)(\forall e)\,[B(cad) \wedge ab \equiv ad \wedge (a \neq c \wedge B(cae) \wedge ab \equiv ae \rightarrow d = e)],$

A 2.2.5 $B(abc) \wedge (B(ade) \vee B(aed)) \wedge ab \equiv ad \wedge ac \equiv ae \rightarrow B(ade) \wedge bc \equiv de,$

A 2.2.6 $a \neq b \wedge ac \equiv ad \wedge bc \equiv bd \wedge B(abe) \rightarrow ec \equiv ed,$

A 2.2.7 $ab \equiv ad \wedge ((B(abc) \wedge B(ade)) \vee (B(cab) \wedge B(ead)) \wedge ac \equiv ae \rightarrow dc \equiv be,$

A 2.2.8 $(\forall ab)(\exists c)\,[B(acb) \wedge ca \equiv cb],$

A 2.2.9 $(\forall abc)(\exists d)\,[\neg L(abc) \rightarrow da \equiv db \wedge db \equiv dc],$

A 2.2.10 $(\forall abcd)(\exists e)\,[B(abc) \rightarrow B(dbe) \wedge ae \equiv ac].$

2.2.1 About the axioms

A2.2.4 is a special case of TARSKI's axiom of segment construction (or of segment transport), which is A1.1.4, and was first considered by J. F. RIGBY [61], [62], [63]; A2.2.6, A2.2.7 are special cases of TARSKI's 'five-segment axiom' A1.1.5; A2.2.5, A2.2.7 are due to RIGBY [63] (A2.2.5 is used in Proposition 2 in Book I of EUCLID's *Elements*, where it is derived from 'Common notion 3') and A2.2.6 is due to H. G. FORDER [24]. A2.2.8 states that every segment has a midpoint, A2.2.9 states that the centre of the circumcircle exists for all triangles; it is a form of Euclid's parallel postulate; A2.2.10 is the circle axiom making a statement similar to that of A1.1.12.

There are some notable axioms missing from our axiom system, namely the PASCH axiom and the transitivity axiom for congruence (A1.1.2), which has been replaced with the three special cases listed in A2.2.3.

The fact that we could omit the PASCH axiom is due to W. SZMIELEW, who proved in [85] that {A1.1.1-A1.1.5, A1.1.8, A1.1.9, A2.1.7, A2.1.8, A2.2.8-A2.2.10} is an axiom system for $\mathcal{E}_2'$. We shall thus prove

Theorem 2.2.1 $Cn(\Upsilon) = \mathcal{E}_2'$.

Our task is to prove that A1.1.2 is a consequence of Υ[2], for afterwards we can prove, as shown by J. F. RIGBY in [61] and [63, (p. 17, 18)], that A1.1.4 is a consequence of Υ, and so is A1.1.5, as shown in [61, p.180].

We now turn to the proof of A1.1.2.

2.2.2 The proof

First, we establish A1.1.6. Suppose $B(aba)$. By A2.2.10 $(\exists e)\,[B(aba) \to B(xbe) \wedge ae \equiv aa]$, and by A1.1.3 $e = a$, so we have, for all x, $B(aba) \to B(xba)$. Let x, y, z be as in A1.1.8, i. e. such that $\neg L(xyz)$. If $a \neq b$, then, since $B(xba) \wedge B(yba) \wedge B(zba)$, we should get $L(xyz)$ (by A2.1.7, A2.1.8, A2.2.1), a contradiction. Therefore $a = b$ and we have proved that A1.1.6 is true.

We now want to prove that there are at least 5 points on each 'line' (i. e. that for $a \neq b$ there are three different points x, y, z, that are different from a and b, such that $L(abx) \wedge L(aby) \wedge L(abz)$). Let $a \neq b$. By A2.2.8 $(\exists x)\,[B(axb) \wedge xa \equiv xb]$, and x cannot be $= a$ or $= b$ as this would imply $a = b$ (by A2.2.2). By A2.2.4 $(\exists y)\,[B(aby) \wedge ba \equiv by]$, and y cannot be $= a$ (as this would imply $a = b$, by A1.1.6) or $= b$ (as this would imply $a = b$, by A2.2.2). By A2.2.4 $(\exists z)\,[B(baz) \wedge ab \equiv az]$, and, for the same reason as above, $z \neq a$, $z \neq b$. So we have found three points x, y, z on the line determined by a and b, and they are all different from a and b. We now want to prove that they are three different points. By A2.1.7 we get $B(yba)$, $B(bxa)$, $B(zab)$, and we deduce from the first two that $B(ybx)$ (by A2.1.8), hence $x \neq y$ (by A1.1.6). From $B(zab)$ and $B(axb)$ we deduce that $B(zax)$, hence $x \neq z$ (by A1.1.6). Suppose $y = z$. From $B(aby)$ and $B(baz)$ (i. e. $B(bay)$) we deduce that $B(aba)$ (by A2.1.8), a contradiction (by A1.1.6). This proves that x, y, z are three distinct points, and hence every line contains at least 5 points.

[2] In the proof of Theorem 2.2.1, we shall call 'true' sentences that are consequences of Υ.

W. Szmielew has proved in [88, Theorem 7.2.7] that, if there are at least 5 points on each 'line', then a relation B that satisfies A2.1.7, A2.1.8, A2.2.1, A1.1.6, satisfies all universal properties of the order relation on a line (i. e. the universal properties of a linear order). From here on, we shall use this fact liberally without mentioning its use.

Let $Hl(abc)$ stand for $B(abc) \vee B(acb)$, i. e. having the intuitive meaning 'c lies on the halfline $\overrightarrow{ab}$'.

In order to prove that

$$bb \equiv dd, \tag{2.1}$$

note that, by A2.2.8, $(\exists a)\, ab \equiv ad$ and that the antecedent of A2.2.5 becomes true if we take $c = b$ and $e = d$. Its consequent gives (2.1).

Since, by A1.1.1, we have $ab \equiv ba \wedge ba \equiv ab$, we deduce from A2.2.3 that

$$ab \equiv ab. \tag{2.2}$$

$$a \neq b \wedge B(abc) \wedge B(abd) \wedge bc \equiv bd \rightarrow c = d \quad \text{(by (2.2), A2.2.4)} \tag{2.3}$$

We now prove that

$$Hl(abb') \wedge ab \equiv ab' \rightarrow b = b'. \tag{2.4}$$

Suppose $B(abb')$ and $a \neq b'$. By A2.2.4 $(\exists c)\, [B(b'ac) \wedge ab' \equiv ac]$. Since $B(b'ac)$ and $B(abb')$, we also have $B(cab)$, so since we also have $B(cab')$ and $ab \equiv ab'$, we conclude that $b = b'$ (by (2.3)). The same conclusion follows if we assume $B(ab'b)$ and $a \neq b$. If $a = b$ or $a = b'$, we deduce $b = b'$ from A2.2.2. This proves (2.4).

Let $a \neq b$. By A2.2.4 $(\exists e)\, [B(cae) \wedge ab \equiv ae]$ and $(\exists d)\, [B(ead) \wedge ae \equiv ad]$. Since $B(cae) \wedge B(ead) \rightarrow Hl(adc)$ (since $a \neq e$ (by A2.2.2)) and $ab \equiv ae \wedge ae \equiv ad \rightarrow ab \equiv ad$ (by A2.2.3(i)), we have

$$(\forall abc)(\exists d)\, [Hl(adc) \wedge ab \equiv ad], \tag{2.5}$$

which is true for $a = b$ as well (with $d = a$).

Let $ab \equiv ac$. We get $(\exists b')\, [Hl(abb') \wedge ac \equiv ab']$ (by (2.5)) and $ab \equiv ac \wedge ac \equiv ab' \rightarrow ab \equiv ab'$ (by A2.2.3(i)), and hence $b = b'$ (by (2.4)), i. e. $ac \equiv ab$, which proves

$$ab \equiv ac \rightarrow ac \equiv ab. \tag{2.6}$$

Let $B(abc) \wedge B(ab'c') \wedge ab \equiv ab' \wedge bc \equiv b'c'$. We get $(\exists d)\, [Hl(adb) \wedge ac' \equiv ad]$ (by (2.5)), $ab' \equiv ab$ (by (2.6), hence $B(abd) \wedge b'c' \equiv bd$ (by A2.2.5, since $ac' \equiv ad \wedge ab' \equiv ab \wedge B(ab'c') \wedge Hl(abd)$). We conclude that $bc \equiv bd$ (by A2.2.3(iii), since $bc \equiv b'c' \wedge b'c' \equiv bd$), so we have $B(abc) \wedge B(abd) \wedge bc \equiv bd$, hence $c = d$ (by (2.3)), which in turn implies $ac' \equiv ac$, whence $ac \equiv ac'$ (by (2.6)), i. e.

$$B(abc) \wedge B(ab'c') \wedge ab \equiv ab' \wedge bc \equiv b'c' \rightarrow ac \equiv ac'. \tag{2.7}$$

Let $oa \equiv oc \wedge oc \equiv oe \wedge ab \equiv cd \wedge cd \equiv ef$. We want to conclude that $ab \equiv ef$. According to A2.2.4 we have $(\exists b')\, [B(oab') \wedge ab \equiv ab']$, $(\exists d')\, [B(ocd') \wedge cd \equiv cd']$, $(\exists f')\, [B(oef') \wedge ef \equiv ef']$. Using (2.6) we get $ab' \equiv ab$, which, together with $ab \equiv cd$, implies $ab' \equiv cd$ (by A2.2.3(ii)), which in turn, together with $cd \equiv cd'$ implies $ab' \equiv cd'$ (by A2.2.3(i)). From (2.6) we get $cd' \equiv cd$, which, together with $cd \equiv ef$, implies $cd' \equiv ef$ (by A2.2.3(ii)), which, together with

$ef \equiv ef'$ implies $cd' \equiv ef'$ (A2.2.3(i)). So far we have proved that $B(oab) \wedge B(ocd') \wedge oa \equiv oc \wedge ab' \equiv cd'$ and $B(ocd') \wedge B(oef') \wedge oc \equiv oe \wedge cd' \equiv ef'$. From the former we conclude $ob' \equiv od'$, from the latter $od' \equiv of'$ (both by (2.7)). From $oa \equiv oc$, $oc \equiv oe$ and $ob' \equiv od'$, $od' \equiv of'$ we get $oa \equiv oe$ and $ob' \equiv of'$ respectively (both by A2.2.3(i)), and from these, together with $B(oab') \wedge B(oef')$, we get $ab' \equiv ef'$ (by A2.2.5). From $ab \equiv ab'$ and $ab' \wedge ef'$ we get $ab \equiv ef'$ (by A2.2.3(ii)), and since we also have $ef' \equiv ef$ (by (2.6)), we get $ab \equiv ef$ (by A2.2.3(i)). This proves

$$oa \equiv oc \wedge oc \equiv oe \wedge ab \equiv cd \wedge cd \equiv ef \rightarrow ab \equiv ef \tag{2.8}$$

We are now ready to prove

$$ab \equiv cd \wedge cd \equiv ef \rightarrow ab \equiv ef. \tag{2.9}$$

If a, c, e are three non-collinear points (i. e. if $\neg L(ace)$), then, by A2.2.9, $(\exists o)\,[oa \equiv oc \wedge oc \equiv oe]$, so A1.1.2 is true, by (2.8). If $a = c$ or $c = e$ or $e = a$, then A1.1.2 is just A2.2.3(ii) or A2.2.3(i) or A2.2.3(iii) respectively. If $e = f$, then $c = d$ and $a = b$ (by A2.2.2), hence A1.1.2 is true by (2.1). Let now a, c, e be different and collinear, i. e. $L(ace)$, and $e \neq f$. Then $(\exists x)\,\neg L(xec)$ (by A1.1.8), and $(\exists f')\,[B(xef') \wedge ef \equiv ef']$ (by A2.2.4), hence $ef' \equiv ef$ (by (2.6)). Since $cd \equiv ef$ and $ef \equiv ef'$, we have $cd \equiv ef'$ (by A2.2.3(i)). Since c, e, f, as well as a, e, f', as well as a, c, f' are three non-collinear points, we can use (2.8) to deduce: from $cd \equiv ef'$, $ef' \equiv f'e$ that $cd \equiv f'e$; from $ab \equiv cd$ and $cd \equiv f'e$ that $ab \equiv f'e$; from $ab \equiv f'e$ and $f'e \equiv ef'$ that $ab \equiv ef'$. We finally deduce from A2.2.3 that, since $ab \equiv ef'$ and $ef' \equiv ef$, we must have $ab \equiv ef$. This proves (2.9).

Since, with the aid of (2.9) we can prove (cf. [63, p.18]) that $ab \equiv cd \rightarrow cd \equiv ab$, from here on, the methods of ([61]) and ([63]) can be used to prove A1.1.4 (cf. [63, p.17,18]) and A1.1.5 (cf. [61, p.180]), so all of W. Szmielew's axioms {A1.1.1-A1.1.5, A1.1.8, A1.1.9, A2.1.7, A2.1.8, A2.2.8-A2.2.10} are true. This proves our main result for the classical axiomatization of Euclidean geometry in $\mathrm{L}_{B\equiv}$, namely Theorem 2.2.1.

2.2.3 The simplicity of Υ

Pourqoui discuter sur les quatre éléments
et les cinq facultés, adolescent?
Qu'importe qu'il y ait une
ou cent énigmes?
Omar Khayyâm, *Robâ'i.*

We have thus seen that $\mathcal{E}_2'$ can be axiomatized by axioms that contain at most 5 variables, and we now show that this is best possible, i. e. that there is no axiom system all of whose axioms contain at most 4 variables.

Let $\mathcal{T} \overset{\text{def}}{=} Cn(\{\varphi \,|\, \varphi \in \mathcal{E}_2' \cap \mathrm{L}_4, \varphi \text{ is written in prenex form}\})$, where L_4 stands for the language that contains the same symbols as $\mathrm{L}_{B\equiv}$, except that there are not countably many but only 4 individual variables.

The idea of the proof is to show that $\mathcal{T}$ is a subtheory of a certain plane geometry in which the congruence relation is not transitive.

The model for this geometry with a non-transitive congruence relation is the plane over the field of real numbers, with the usual affine and order structures, but with a congruence relation that is strictly included in the usual congruence relation of the Cartesian plane over the reals.

Let $\mathfrak{M} = \langle \mathbb{R}\times\mathbb{R}, \mathbf{B}_{\mathbb{R}}, \equiv_{\mathfrak{M}}\rangle$, where $\mathbf{a}_1\mathbf{a}_2 \equiv_{\mathfrak{M}} \mathbf{a}_3\mathbf{a}_4$ iff $\mathbf{a}_1\mathbf{a}_2 \equiv_{\mathbb{R}} \mathbf{a}_3\mathbf{a}_4$ and one of the following is true

(i) $\mathbf{a}_i\mathbf{a}_j$ is parallel to $\mathbf{a}_k\mathbf{a}_l$, for some $\{i,j,k,l\} = \{1,2,3,4\}$;

(ii) $\mathbf{L}_{\mathbb{R}}(\mathbf{a}_i\mathbf{a}_j\mathbf{a}_k)$ for some i,j,k with $i \neq j \wedge j \neq k \wedge k \neq i$ and $i,j,k \in \{1,2,3,4\}$;

(iii) $\mathbf{a}_i\mathbf{a}_j \equiv_{\mathbb{R}} \mathbf{a}_i\mathbf{a}_k$ for some i,j,k with $i \neq j \wedge j \neq k \wedge k \neq i$ and $i,j,k \in \{1,2,3,4\}$;

(iv) the measure of one of the angles between $\mathbf{a}_1\mathbf{a}_2$ and $\mathbf{a}_3\mathbf{a}_4$ is $\frac{\pi}{n}$ for some $n \in \mathbb{N}\setminus\{0\}$.

Let $\mathcal{C} = Th_{\mathrm{L}_{B\equiv}}(\mathfrak{M}) \cap \mathcal{E}_2'$.

Theorem 2.2.2 *$\mathcal{T} \subseteq \mathcal{C}$.*

Proof. In order to prove that there is no sentence $\sigma \in \mathcal{T}\setminus\mathcal{C}$ we shall use the model-theoretic method of EHRENFEUCHT-FRAÏSSÉ games, as described in [40].

The method given there allows us to prove that a certain sentence $\sigma \in \mathcal{E}_2' \setminus \mathcal{C}$ is not equivalent (with respect to $\mathcal{C}$) to a sentence in prenex form with 4 quantifiers.[3] of a given prefix type. For each prefix type with 4 quantifiers the game method allows us to prove that *no* sentence $\sigma \in \mathcal{E}_2' \setminus \mathcal{C}$ is equivalent to one of that particular type.

Let σ be any sentence in $\mathcal{E}_2' \setminus \mathcal{C}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of $\mathcal{C}$, such that $\mathfrak{A} \models \sigma$, but $\mathfrak{B} \not\models \sigma$ (for example, let $\mathfrak{A}$ be $\mathfrak{C}_2(\mathbb{R})$ and let $\mathfrak{B}$ be the plane $\mathfrak{M}$ used to define $\mathcal{C}$). The EHRENFEUCHT-FRAÏSSÉ game to be used in order to prove that σ is not $\mathcal{C}$-equivalent to a sentence with a certain prefix containing 4 quantifiers can be described as follows:

In this game, there are two players, I and II, that alternate in making choices from the two models $u(\mathfrak{A})$ and $u(\mathfrak{B})$, (it depends on the prefix which set a player is supposed to chose from at the n^{th} move; a universal quantifier in the n^{th} position forces I to choose from $u(\mathfrak{B})$, an existential one forces I to choose from $u(\mathfrak{A})$)). The choice of I at the n^{th} move will be denoted by $\mathbf{x}_n$, the choice of II at the n^{th} move by $\mathbf{y}_n$. Let $\{\mathbf{a}_n\} = \{\mathbf{x}_n, \mathbf{y}_n\} \cap u(\mathfrak{A})$ and $\{\mathbf{b}_n\} = \{\mathbf{x}_n, \mathbf{y}_n\} \cap u(\mathfrak{B})$. Player II wins the game, which in our case consists of 4 moves, if at the end of the game the function f, defined by $f(\mathbf{a}_n) = \mathbf{b}_n$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The fact that σ is not $\mathcal{C}$-equivalent to a sentence with prefix containing 4 quantifiers is equivalent to the existence of a winning strategy for II in the corresponding game.

Let $\mathfrak{A}$ be $\mathfrak{C}_2(\mathbb{R})$ and let $\mathfrak{B}$ be $\mathfrak{M}$. The winning strategy for player II is:
Choose for the first three moves points with coordinates identical to those chosen by I. By abuse of language we shall denote these first three moves by the same letters.

In the fourth move, if

(i) II has to chose from $u(\mathfrak{B})$,

(ii) the first three points chosen are not collinear,

(iii) I has made the fourth choice such that the distance from $\mathbf{x}_4$ to $\mathbf{v}$, one of the vertices of the triangle Δ formed by the first three ('common' choices), is congruent (in the standard plane) to $\mathbf{ab}$, that side of the triangle which does not pass through $\mathbf{v}$, but the distance from

[3] The fact that there is a winning strategy for the second player for 4-moves in this game implies more than just the non-existence of a sentence in *prenex form* with 4 quantifiers, cf. [40].

$\mathbf{x}_4$ to any other vertex of the triangle is not congruent to any other side of Δ,
then II chooses $\mathbf{y}_4$ such that
(1) $\mathbf{vy}_4$ is congruent to $\mathbf{ab}$ (in the standard plane),
(2) the angle between $\mathbf{vy}_4$ and $\mathbf{ab}$ is $\frac{\pi}{n}$ for some $n \in \mathbb{N} \setminus \{0\}$,
(3) none of the distances $\mathbf{y}_4\mathbf{a}$ or $\mathbf{y}_4\mathbf{b}$ is equal to any of the sides of the triangle Δ.

Otherwise, i. e. if in the fourth move we are not in the situation described by (i), (ii) and (iii), choose the fourth point to have the same coordinates as the point chosen by player I. □

From Theorem 2.2.1 and Theorem 2.2.2 we conclude that

Theorem 2.2.3 $sd(\mathcal{E}_2') = 5$.

2.3 Simple axiom systems for $\mathcal{E}_n'$

Therefore, there is neither an existent nor a non-existent,
neither the characterized nor the characteristics,
neither space nor the five elements similar to space.
Nāgārjuna, *Mūlamadhyamakakārikā.*

Let $\mathcal{E}_n'$ (with $n \geq 2$) be the $\mathrm{L}_{B\equiv}$-theory common to all n-dimensional Cartesian spaces $\mathfrak{C}_\mathfrak{n}(\mathfrak{F})$ over Euclidean ordered fields F.

D. Scott proved in [70, Th. 3.4] that any sentence φ (in any first-order language with variables to be interpreted as 'points', and with primitive notions that are 'geometric' (in the sense that they are invariant under isometries)), that contains at most $n+1$ variables and that is true in the n-dimensional Cartesian space over, say $\mathbb{R}$, is also true in all m-dimensional Cartesian spaces over $\mathbb{R}$, for all $m \geq n$. This implies that $sd(\mathcal{E}_n') \geq n+2$ for all $n \geq 2$. We shall prove that $sd(\mathcal{E}_n') = n+2$ for all $n \geq 3$. In particular, 3-dimensional Euclidean geometry has Simplicity degree 5. A simple axiom system for $\mathcal{E}_n'$ (with $n \geq 2$) consists of $(\Upsilon \setminus \{$A1.1.8, A1.1.9$\}) \cup \{$A2.3.1(n), $\neg$ A2.3.1(n+1)$\}$, where

A 2.3.1 (n) $(\exists a_1 a_2 \ldots a_{n+1}) \bigwedge_{p<q, r<s} a_p a_q \equiv a_r a_s$.

We have thus proved

Theorem 2.3.1 *$Sd([\mathcal{E}_n']) = n+2$ for all $n \geq 3$ and the axiom system $(\Upsilon \setminus \{$A1.1.8, A1.1.9$\}) \cup \{$A2.3.1(n), $\neg$ A2.3.1(n+1)$\}$ is simple, regardless of language (cf. Simplicity Criterion 1.2.4).*

We shall see in §3.7 that $Sd([\mathcal{E}_n]') = n+2$ for $n = 2$ as well.

2.4 A simple axiom system for $\mathcal{E}_2$

Oṃ Ma Ṇi Pad Me Hûm
The six-syllabled mantra of *Avalokiteśvara.*

In [50] we sketched a proof that $sd(\mathcal{E}_2) = 6$ and a simple axiom system is $\Xi = \{$A1.1.6, $\Upsilon \setminus \{$A2.2.8, A2.2.10$\}$, A1.1.7$\}$.

Ξ is an axiom system for $\mathcal{E}_2$, because our proof of A1.1.2 in §2.2.2 used A2.2.10 only to prove A1.1.6, which is now an axiom of Ξ , and A2.2.8 only to prove that there are at least 5 points on every line, a fact that can be proved by using only A2.2.4, so we can deduce A1.1.2 from Ξ as well. The proof of the existence of midpoints (i. e. of A2.2.8) can be now carried out exactly as in [63], since we now have Pasch's axiom at our disposal. We have thus proved all the axioms that were used in [63], and the axiom system proposed there is an axiom system for $\mathcal{E}_2$.

The reason for the impossibility of expressing the axioms of $\mathcal{E}_2$, with only 5 variables lies with the PASCH axiom. To make this statement precise, we introduce the PASCH-free geometry $\mathcal{E}_2^- \stackrel{\text{def}}{=} Cn($A1.1.6, $\Upsilon \setminus \{$A2.2.10$\})$, for which we have the following

Representation Theorem 2.4.1 (SZCZERBA - SZMIELEW **[84]**) *$\mathfrak{M} \in \mathfrak{Mod}(\mathcal{E}_2^-)$ iff $\mathfrak{M} \simeq \langle \mathfrak{F}_s \times \mathfrak{F}_s, \equiv_{\mathfrak{F}_s}, \mathbf{B}_{\mathfrak{F}_s}\rangle$, for some formally real and Pythagorean semi-ordered field F_s, with $\mathbf{B}_{F_s}(\mathbf{abc})$ iff $|\sqrt{\|\mathbf{a}-\mathbf{b}\|}\,|_s + |\sqrt{\|\mathbf{b}-\mathbf{c}\|}\,|_s = |\sqrt{\|\mathbf{c}-\mathbf{a}\|}\,|_s$.*

A subset P of a field F is called a semi-positive cone for F (i. e. $1 \in P$, $P \cup -P = F$, $P \cap -P = \{0\}$, $P + P = P$, but not necessarily $P \cdot P \subseteq P$). A field $\langle F, \leq_s\rangle$, for which we may write simply F_s, is called semi-ordered if F is a field, P a semi-positive cone of F and $x \leq_s y$ iff $y - x \in P$. The semi-absolute value $|x|_s$ is defined to be x if $x \in P$ and $-x$ if $-x \in P$.

Since there are, as shown in [82] and [59], properly semi-ordered (i. e. the semi-positive cone P does not satisfy $P \cdot P \subseteq P$) formally real and Pythagorean fields (even $\mathbb{R}$ can be properly semi-ordered, assuming the axiom of choice, as shown in [82]), we have $\mathcal{E}_2^- \subset \mathcal{E}_2$.

The following axioms, each requiring only 5 variables, when added to $\mathcal{E}_2^-$, are equivalent to the algebraic statement that the semi-order of the coordinate field is quadratic, i. e. that the semi-positive cone of F satisfies $P \cdot F^2 = P$.

A 2.4.1 $B(acb) \wedge B(adb) \wedge ae \equiv ad \wedge be \wedge bc \wedge c \neq d \rightarrow B(acd)$,

A 2.4.2 $d \neq a \wedge B(bac) \wedge ab \equiv ac \wedge db \equiv dc \wedge ae \equiv bd \wedge (B(ade) \vee B(aed)) \rightarrow B(ade)$,

Axiom A2.4.1 is the triangle inequality, and A2.4.2 states that the hypotenuse is greater than the other sides of a right triangle.

The fact that A2.4.1 and A2.4.2 are equivalent with the fact that the semi-order of the coordinate field is quadratic, was proved in [59, (Satz 2.3, p.89)]. Let $\mathcal{E}_2^{-q} \stackrel{\text{def}}{=} Cn(\mathcal{E}_2^- \cup \{$A2.4.1$\}) = Cn(\mathcal{E}_2^- \cup \{$A2.4.2$\})$ Since there are quadratically semi-ordered fields which are not ordered (cf. [31], [59]), we have $\mathcal{E}_2^{-q} \subset \mathcal{E}_2$.

We have sketched in [50] a proof of the fact that there is no sentence $\sigma \in \mathcal{E}_2 \setminus \mathcal{E}_2^{-q}$ which, when written in prenex form has only 5 variables. Since our proof that $\mathcal{E}_2'$ cannot be axiomatized by axioms in L_4 extends to $\mathcal{E}_2^-$ and $\mathcal{E}_2^{-q}$, we have thus proved that

Theorem 2.4.1
(i) $sd(\mathcal{E}_2) = 6$,
(ii) $sd(\mathcal{E}_2^-) = sd(\mathcal{E}_2^{-q}) = 5$.

The Sd of $[\mathcal{E}_2]$ is not 6, since we can enlarge the language $\mathrm{L}_{B\equiv}$ with the quaternary predicate symbol Int (standing for $Int(abcd)$ iff 'the segments ab and cd intersect') and get an axiom system all of whose axioms have at most 5 variables, by adding to the axioms for $\mathcal{E}_2^-$ the sentences

$$(\forall abcd)(\exists e)\,[Int(abcd) \rightarrow B(aeb) \wedge B(ced)];\;\; B(aeb) \wedge B(ced) \rightarrow Int(abcd), \tag{2.10}$$

$$\neg L(abc) \wedge B(aeb) \wedge B(bcd) \rightarrow Int(aced). \tag{2.11}$$

We shall see in §3.7 that $\mathrm{Sd}([\mathcal{E}_2]) = 4$ as well.

It is worth noting that Theorem 2.4.1 tells us that the statement of the PASCH axiom requires the greatest number of variables among all the axioms of ordered Cartesian planes with free mobility, in other words that it is, in the $\mathrm{L}_{B\equiv}$-setting, the most complicated of all the axioms.[4] It is also interesting to note that it was the last one among all the axioms for Euclidean geometry to be 'discovered' (in 1882 by M. PASCH) and the last one to be proven independent of the others in a suitable axiom system (by L. W. SZCZERBA [82] in 1970, while we still do not know whether it is independent in TARSKI's axiom system (stated in §1.1)). R. SCHNABEL [65, p. 14] commented on it as follows;

> Was nun das Problem des Maßes angeht, so verstehe ich darunter die Frage, ob man die Aspekte der Anordnung und Stetigkeit bei einer ersten Begründung der Geometrie berücksichtigt oder nicht. Die Frage ist naheliegend, weil diese Aspekte des Räumlichen zu einer erheblichen *Kompliziertheit*[5] der Axiomensysteme führen, sie entziehen sich zunächst einer glatten Beschreibung. In der Tat sind diese Aspekte wohl die verborgensten in der Raumanschauung, die am wenigsten bewußten, und damit zugleich die anschaulich primitivsten und gebräuchlichsten.

2.5 A small step toward semantical simplicity

> Der Weltgeist will nicht fesseln uns und engen,
> Er will uns Stuf' um Stufe heben, weiten.
> Hermann Hesse, *Stufen.*

We present in this section a contribution towards a semantically simple axiom system for Euclidean geometry, by splitting the parallel axiom.

Whenever we meet in the foundations of geometry a situation where, as soon as we have a certain part $\mathcal{T}$ of the desired axiom system, and an axiom α turns out to be stronger than another axiom β, where neither α nor β are in $Cn(\mathcal{T})$, the question naturally arises of whether there is a *geometric statement* β', which is weaker than α, but which, together with β, is equivalent to α.

From a logical point of view, there is no problem in finding such a β' since we are asking for: if

$$\mathcal{T} \vdash \alpha \rightarrow \beta,\; \mathcal{T} \nvdash \beta,\; \text{and } \mathcal{T},\, \beta \nvdash \alpha,$$

[4] It should also be noted that the circle axiom may also be expressed by a 4-variable sentence, namely $(\forall abc)(\exists d)\,[B(abc) \rightarrow da \equiv db \wedge ad \equiv ac]$, i. e. adding this axiom to the axiom system for $\mathcal{E}_2^-$ we get an axiom system for $\mathcal{E}_2'$.

[5] My italics.

then find β', such that

$$\mathcal{T} \vdash \alpha \leftrightarrow \beta \wedge \beta' \text{ and } \mathcal{T},\, \beta' \nvdash \alpha.$$

Such a β' is obviously $\beta \to \alpha$. However, the logical formulation of this problem omits the requirement that β' should be a 'geometric statement'.

By 'geometric statement' we mean a statement that one would consider adopting in a deductive development of geometry as either an axiom or a theorem. We admit that we did not define anything by having stated the above 'meaning' of a 'geometric statement', but on the other hand we cannot pretend to decide a priori of what form all the geometrically meaningful statements should be.

Since most axiomatizations of Euclidean geometry (like, e. g. the axiom system in [72], that has been reproduced in chapter 1, as well as Υ, with A2.2.4 split into two axioms, one stating the existence of the transported segment-endpoint, and a universal statement stating the uniqueness of the transported segment-endpoint) consist of universal-existential-sentences ($\forall\exists$-sentences; i. e. all universal quantifiers ($\forall$) precede all existential quantifiers ($\exists$)), one would be inclined (in case one does not agree with the above-mentioned methodological remark) to define 'geometric statements' as $\forall\exists$-sentences in the language in which one intends to axiomatize geometry. In that case, we should consider on the one hand '(DESARGUES' theorem) *or* (There is a rectangle) (R)', clearly undesirable as an axiom, to be a 'geometric statement', and on the other hand, sentences like the local form of the parallel axiom, i. e. 'There is a point P and a line g, such that there is at most one parallel to g through P' would not be deemed 'geometric'.

The problem of the meaningful geometric splitting of axioms was first successfully solved by W. SZMIELEW in [86], where she proposes a geometric statement β' (which states : 'For three different points A, B and C, the perpendicular bisector of AB intersects one of the circles that have AC respectively BC as diameters'), for $\mathcal{T} = \mathcal{E}_2^-$, $\alpha =$ A1.1.12 and $\beta =$ A1.1.7.

We call such a β' a *missing link* between β and α with respect to $\mathcal{T}$ and say that α was *split* with respect to $\mathcal{T}$ in β and β'.

We have given some reasons why we should be interested in non-trivially splitting axioms in §1.2. It is through splitting that one hopes to eventually arrive at a semantically simple axiom system.

M. DEHN gave in [17] examples of non-Archimedean planes with Euclidean metric, in which the Euclidean parallel axiom (*Par*, which may be stated in various forms, e. g. as A2.2.9, A2.1.4, A1.1.10) is not valid, thereby proving that R is weaker than the latter with respect to plane absolute geometry ($\mathcal{A}_2$, which is $Cn(\text{A1.1.1} - \text{A1.1.9}) = Cn(\Xi \setminus \{\text{A2.2.9}\})$). F. BACHMANN proposed in [9] a further weakening of the axiom R, namely *Every quadrilateral with three right angles closes* (A) (see Fig. 2.3). In $\mathrm{L}_{B\equiv}$, axiom A would be expressed as

A 2.5.1 $(\forall abcxyztu)(\exists d)\,[a \neq b \wedge b \neq c \wedge B(xba) \wedge ba \equiv bx \wedge B(ycb) \wedge cb \equiv cy \wedge zb \equiv zy$
$\wedge\, z \neq c \wedge B(tab) \wedge at \equiv ab \wedge u \neq a \wedge ub \equiv ut \to L(czd) \wedge L(aud)]$.

F. BACHMANN has proved in [10, S. 176] that A (which he calls *Lotschnittaxiom*) is equivalent to the following specialization of TARSKI's choice of the Euclidean parallel postulate A1.1.10: *Through every point in the interior of a right angle, there is a line intersecting the sides of that angle.*

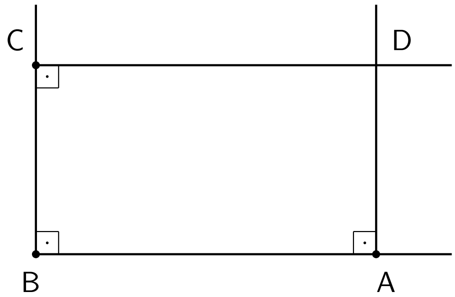

Figure 2.3: The *Lotschnittaxiom* (A)

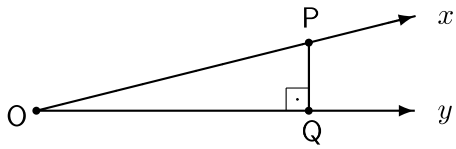

Figure 2.4: ARISTOTLE's Axiom (Ar)

F. BACHMANN proved in [9] that we have the following strict implications in $\mathcal{A}_2$

$$Par \to R \to A, \quad \text{but} \quad A \not\to R \not\to Par. \tag{2.12}$$

It would therefore be desirable to find two missing links, β_1 and β_2, β_1 from A to R and β_2 from R to Par, such that $\mathcal{A}_2 \nvdash A \wedge \beta_2 \to Par$, and to hereby split the Euclidean parallel postulate into A, β_1 und β_2. This problem remains open.

We shall, however, find a missing link between A and Par with respect to $\mathcal{A}_2$. The missing link is ARISTOTLE's axiom (Ar), whose importance in the foundations of geometry (with special emphasis on the hyperbolic case) was studied by M. J. GREENBERG [27].

It states that (see Fig. 2.4) the lengths of the segments, whose endpoints lie on the sides of any acute angle and which are perpendicular to one of the sides, grow indefinitely, i. e. can be made longer than any given segment. The exact $\mathrm{L}_{B\equiv}$-statement is

A 2.5.2 $(\forall oxyy'a)(\exists pqq'z)\,[B(y'oy) \wedge oy \equiv oy' \wedge a \neq o \wedge ay' \equiv ay \wedge B(axy)$
$\to (B(oxp) \vee B(opx)) \wedge (B(oyq) \vee B(oqy)) \wedge B(oqq')$
$\wedge\, qo \equiv qq' \wedge po \equiv pq' \wedge pq \equiv oz \wedge B(oyz)]$.

2.5.1 Algebraic description of H-planes that satisfy the *Lotschnittaxiom*

Since we could not prove synthetically that

$$\mathcal{A}_2 \vdash A \wedge Ar \to Par, \tag{2.13}$$

we shall have to use the algebraic characterization of models of plane absolute geometry $\mathcal{A}$ (to be called H-planes from now on) given by W. PEJAS (cf. [58], [34], [9]).

We shall give a brief summary of the results in [9].

Let K be a field and k an element of K. By a *projective-metric plane* $\mathbf{A}(K,k)$ (cf. [34, p.215]) over the field K with the *metric constant* k we mean the projective plane $\mathbf{P}(K)$ over the field K, for whose points of the form $(x,y,1)$ we shall write (x,y), together with a notion of orthogonality, the lines $[u,v,w]$ and $[u',v',w']$ being orthogonal iff

$$uu' + vv' + kww' = 0.$$

If K is an ordered field, then one can order the affine part $\mathbf{A}(K)$, which consists of all the points in $\mathbf{P}(K)$ that do not lie on the line $[0,0,1]$ in the usual way.

The algebraic characterization of the H-planes consists in specifying a point-set E of an affine-metrc plane $\mathbf{A}(K,k)$, which is the universe of the H-plane. Since E will always lie in $\mathbf{A}(K)$, the H-plane will inherit the ordering relation B from $\mathbf{A}(K)$. The congruence of two segments $\mathbf{ab}$ and $\mathbf{cd}$ will be given, if $E \subset \mathbf{A}(K,0)$, by the usual Euclidean formula

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = (c_1 - d_1)^2 + (c_2 - d_2)^2$$

and, if $E \subset \mathbf{A}(K,k)$ with $k \neq 0$, by

$$\frac{F(\mathbf{a},\mathbf{b})^2}{Q(\mathbf{a})Q(\mathbf{b})} = \frac{F(\mathbf{c},\mathbf{d})^2}{Q(\mathbf{c})Q(\mathbf{d})}, \tag{2.14}$$

where

$$F(\mathbf{x},\mathbf{y}) = k(x_1y_1 + x_2y_2) + 1,\ Q(\mathbf{x}) = F(\mathbf{x},\mathbf{x}),\ \text{und } \mathbf{x} = (x_1,x_2), \mathbf{y} = (y_1,y_2).$$

In the course of §2.5, let K be an ordered Pythagorean field, R the ring of *finite* elements, i. e. $R = \{x \in K \mid (\exists n \in \mathbf{N})\ |x| < n\}$ and P the ideal of *infinitely small* elements of K, i. e. $P = \{0\} \cup \{x \in K \mid x^{-1} \notin R\}$.

F. Bachmann shows in [9] that the H-planes which satisfy A are of two types, namely

Type I *(corresponds to Theorem 1 in [9])* $E = \{(a,b) \mid a,b \in M\} \subset \mathbf{A}(K,0)$, *where M is an R-module $\neq (0)$;*

Type II *(corresponds toTheorem 2 in [9])* $E = \{(a,b) \mid a,b \in M\} \subset \mathbf{A}(K,k)$ *with $k \neq 0$, where M is an R-module $\neq (0)$ included in $\{a \in K \mid ka^2 \in P\}$, that satisfies the condition*

$$a \in M \Rightarrow ka^2 + 1 \in K^2$$

2.5.2 Aristotle's axiom is not valid in H-planes of type II

We shall now prove that

$$\mathcal{A}_2 \vdash A \wedge Ar \to R. \tag{2.15}$$

M.J. Greenberg ([27, Theorem 1]) proved that the metric constant k of an H-plane that satisfies Ar must be ≤ 0. Therefore, H-planes that satisfy both A and Ar can be only of type I or type II with $k < 0$. We shall prove that

Theorem 2.5.1 *Ar is not valid in H-planes of type II with $k < 0$.*

Proof. Let $k < 0$. Since the metric constant k is determined only up to a quadratic factor $\neq 0$ by the given H-plane, we may assume w. l. o. g. that $k \in K \setminus P$.

Since $M \subseteq \{a \in K \mid ka^2 \in P\}$ we have $M \subseteq P$. In particular, $ab \in M$ if $a, b \in M$, since M is an R-module and $M \subseteq P \subset R$.

Let $p > 0$ be an element of M, and $\overrightarrow{\mathsf{O}x}$ resp. $\overrightarrow{\mathsf{O}y}$ (where $\mathsf{O} = (0,0)$) the halflines in $\mathbf{A}(K,k)$, whose points have the coordinates (m, pm) resp. $(m, 0)$, with $m \in M$, $m > 0$ $(\mathsf{O} = (0,0))$. We claim that every segment, whose endpoints P and Q lie on $\overrightarrow{\mathsf{O}x}$ resp. $\overrightarrow{\mathsf{O}y}$ such that $\mathsf{PQ} \perp \overrightarrow{\mathsf{O}y}$, is shorter than OB, where $\mathsf{B} = (p, 0)$.

If $\mathsf{P} = (m, pm)$, then we must have $\mathsf{Q} = (m, 0)$, because PQ is orthogonal to $\overrightarrow{\mathsf{O}y}$. We transport the segment PQ from O on $\overrightarrow{\mathsf{O}x}$, and obtain thereby a point $\mathsf{C} = (c, 0)$, such that $\mathsf{OC} \equiv \mathsf{PQ}$, i. e. by (2.14),

$$\frac{F((m,pm),(m,0))^2}{Q((m,pm))Q((m,0))} = \frac{F((0,0),(c,0))^2}{Q((0,0))Q((c,0))}. \tag{2.16}$$

From (2.16) we deduce

$$c^2 = \frac{p^2 m^2}{1 + km^2},$$

and hence $0 < c < p$, since m^2 and km^2, and hence also $\frac{m^2}{1+km^2}$, are infinitely small. Therefore C always lies between O and B. □

We conclude that H-planes that satisfy both A and Ar can be only of type I, and those are exactly those H-planes that satisfy R (cf. [58] or [9]). This proves (2.15).

2.5.3 H-planes with Euclidean metric which satisfy Aristotle's axiom

We shall now prove[6] that

$$\mathcal{A}_2 \vdash R \wedge Ar \rightarrow Par. \tag{2.17}$$

Let g be a line and O a point that does not lie on g (see Fig. 2.5). We want to prove that there is only one parallel through O to g. Let A be the footpoint of the perpendicular from O to g, and let h be the perpendicular at O to OA. Let B and C be two points on g such that A lies between B and C, and let B' and C' be the footpoints of the perpendicular from B and C to h. Let now h' be a line that is different from h and that passes through O. We want to prove that h' and g will have to intersect. h' must (as required by the order axioms, in particular by the Pasch axiom) pass either through the interior of the rectangle OACC' or through the interior of the rectangle OABB'. We may assume w. l. o. g. that h' passes through the interior of the rectangle OACC'. Let $\vec{\mathsf{h}}'$ be that halfline of h' which passes through O and goes through the interior of OACC'. If h' is different from OA, then the angle formed by the halflines $\vec{\mathsf{h}}'$ and $\overrightarrow{\mathsf{OC}'}$ is acute, and hence there are, by Ar, two points P and Q on $\vec{\mathsf{h}}'$ resp. $\overrightarrow{\mathsf{OC}'}$, such that $\mathsf{PQ} \perp \overrightarrow{\mathsf{OC}'}$ and PQ is longer than OA. Since the distance between h and g is everywhere the same (as R holds in our plane), the points P and Q have to lie on different

[6] This is essentially Proclus's proof of Euclid's Fifth Postulate.

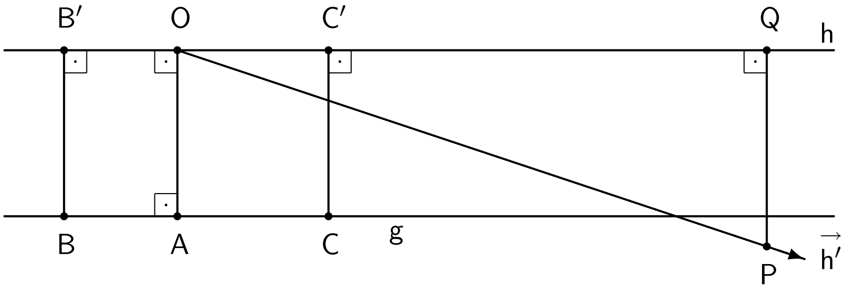

Figure 2.5: $\mathcal{A}_2 \vdash R \wedge Ar \to Par$

sides of g, whence O and P lie on different sides of g. By the PASCH axiom, OP (i. e. $\overrightarrow{\mathsf{h}'}$) intersects g, q. e. d.

2.5.4 $\{A, Ar\}$ is a proper splitting of *Par*

It follows from (2.15) and (2.17) that

$$\mathcal{A}_2 \vdash A \wedge Ar \to Par.$$

Since $\mathcal{A}_2 \vdash Par \to A \wedge Ar$ is well-known, $\mathcal{A}_2 \nvdash A \to Par$ was proved in [9] (cf. (2.12)), and $\mathcal{A}_2 \nvdash Ar \to Par$ was proved in [27, Theorem 2] (Ar is valid in the plane hyperbolic geometry as well), we conclude that $\{A, Ar\}$ is a splitting of Par with respect to $\mathcal{A}_2$.

If $\{A, Ar\}$ were a trivial splitting of Par, then there would exist a sentence δ, such that $\mathcal{A}_2 \vdash A \leftrightarrow Par \vee \delta$ and $\mathcal{A}_2 \vdash Ar \leftrightarrow Par \vee \neg\delta$, or equivalently we should have $\mathcal{A}_2 \vdash A \vee Ar$, which is false, as we shall prove that there are H-planes that satisfy neither A nor Ar.

The H-planes that do not satisfy A are of

Type III *(corresponds to Theorem 3 in [9])*
$E = \{\mathbf{x} \mid Q(\mathbf{x}) > 0, Q(\mathbf{x}) \notin J\} \subset \mathbf{A}(K, k)$ *with* $k < 0$*, where* $J \subseteq P$ *is a prime ideal of* R *that satisfies the condition*

$$ka^2 + 1 > 0, ka^2 + 1 \notin J \Rightarrow ka^2 + 1 \in K^2,$$

with K *satisfying*

$$\{a \in K \mid ka^2 \in R \setminus P\} \neq \emptyset$$

Theorem 2.5.2 *Let* K *be a Euclidean non-Archimedean ordered field,* $k = -1$ *and* $J = P$*. The corresponding* H*-plane of type III satisfies neither* A *nor* Ar*.*

Proof. Let $p \in P$ and $\overrightarrow{\mathsf{O}x}$, $\overrightarrow{\mathsf{O}y}$ be as in Theorem 2.5.1, with the difference that now $m \in K$ and the points on these halflines must be in a set E of type III i. e. for every point $(m, 0)$ on $\overrightarrow{\mathsf{O}y}$ we must have

$$1 - m^2 > 0 \text{ and } 1 - m^2 \notin P. \tag{2.18}$$

We claim that any segment, whose endpoints P and Q lie on $\overrightarrow{\mathsf{O}x}$ resp. $\overrightarrow{\mathsf{O}y}$, such that $\mathsf{PQ} \perp \overrightarrow{\mathsf{O}y}$, is shorter than the segment OB, where

$$\mathsf{B} = (\frac{1}{2}, 0).$$

Let $P \in \overrightarrow{\mathsf{O}x}$, $Q \in \overrightarrow{\mathsf{O}y}$, $\mathsf{PQ} \perp \overrightarrow{\mathsf{O}y}$ and $\mathsf{C} = (c, 0)$ on $\overrightarrow{\mathsf{O}y}$, such that $\mathsf{OC} \equiv \mathsf{PQ}$. Then, as in Theorem 2.5.1,

$$c^2 = \frac{p^2 m^2}{1 - m^2}.$$

Since, according to (2.18), $m^2 < 1$ and $(1 - m^2)^{-1} \in R$, and p was chosen to be in P, c^2, and hence c as well, must be in P, and therefore must be smaller than $\frac{1}{2}$. Therefore the segment OC, and hence PQ, are smaller than OB. □

Theorem 2.5.2 also follows from [27, Theorem 3] .

2.5.5 The *Lotschnittaxiom* as universal statement

F. Bachmann proved in [10, Satz 6] that axiom A is hereditary, i. e. if A holds in an H-plane $\mathfrak{H}$, then it holds in any H-plane $\mathfrak{H}' \subseteq \mathfrak{H}$ as well.

From a logical point of view this amounts to the existence of a universal sentence A', that is logically equivalent with A with respect to $\mathcal{A}_2$, i. e.

$$\mathcal{A}_2 \vdash A \leftrightarrow A'. \tag{2.19}$$

However, none of the three equivalent formulations of axiom A, stated by F. Bachmann in [9], is a universal sentence. We shall now prove that

Theorem 2.5.3 *The universal statement*

$$\begin{aligned} A' \quad & q \neq o \wedge m \neq m' \wedge B(m'om) \wedge mo \equiv mo' \wedge m'o \equiv mo \wedge B(omo') \\ & \wedge qm \equiv qm' \wedge po \equiv po' \wedge po \equiv pr \wedge qo \equiv qr \wedge B(oqr) \wedge pq \equiv pm \wedge pm \equiv op' \\ & \wedge (B(op'o') \vee B(oo'p')) \rightarrow B(op'o') \wedge p' \neq o' \end{aligned}$$

is equivalent to A.

A' states that the base-height in an isosceles triangle, whose base-angles are half-right is smaller than the base (see Fig. 2.6).

Proof. In order to prove (2.19), it is sufficient to prove that A' holds in all H-planes of type I and II but not in those of type III. A' clearly holds in the H-planes of type I, since these have Euclidean metric, and hence the length of the base-height of $\triangle\mathsf{POO}'$ is half the length of the base.

For H-planes of type II and III let $\mathsf{O} = (0, 0)$, let OO' be the x-axis and $\mathsf{P} = (m, m)$ with $m > 0$. Then $\mathsf{M} = (m, 0)$, $\mathsf{O}' = (\frac{2m}{1-km^2}, 0)$ and $\mathsf{P}' = (\frac{m}{\sqrt{1+km^2}}, 0)$[7] We shall prove that the

[7] $\sqrt{1 + km^2}$ stands for the positive solution of $X^2 = 1 + km^2$.

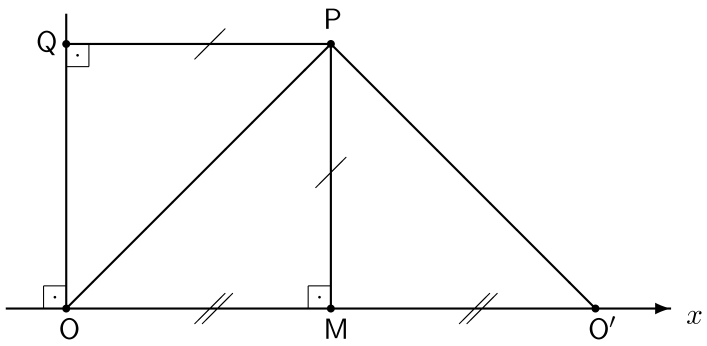

Figure 2.6: Axiom A'

inequality required by A', namely

$$\frac{m}{\sqrt{1+km^2}} \leq \frac{2m}{1-km^2} \tag{2.20}$$

is satisfied in all H-planes of type II, and that there are in all H-planes of type III points $\mathsf{P} = (m, m)$ for which (2.20) is not satisfied. Elementary computations show that (2.20) is equivalent with

$$(km^2)^2 - 6km^2 \leq 3 \tag{2.21}$$

The left hand side of the inequality (2.21) is infinitely small in H-planes of type II, since $M \subset \{a \in K \mid ka^2 \in P\}$, so (2.21) holds in these H-planes.

The requirement that $\mathsf{P} = (m, m) \in E$ implies, for H-planes of type III,

$$2km^2 + 1 > 0 \text{ and } 2km^2 + 1 \notin J. \tag{2.22}$$

Let $x = km^2$. Then (2.21) is equivalent to

$$3 - 2\sqrt{3} \leq x \leq 3 + 2\sqrt{3}$$

and (2.22) will certainly be satisfied if $x > -\frac{1}{2}$ and $x + \frac{1}{2} \notin P$. Since $\{a \in K \mid ka^2 \in R \setminus P\} \neq \emptyset$, let $a_0 \in K$, such that $ka_0^2 \in R \setminus P$. Therefore there are two natural numbers p and q, such that $-\frac{1}{2} + \frac{1}{100} < k(\frac{p}{q}a_0)^2 < 3 - 2\sqrt{3}$. Let now $\mathsf{P} = (m, m)$ with $m = \frac{p}{q}a_0$. P is in E, but (2.21), and hence (2.20) as well, are not satisfied, i. e. in triangle $\triangle\mathsf{POO'}$ the height through P is longer than $\mathsf{OO'}$. □

2.6 A binary relation as primitive notion

più uno, più solo,
battere fondo del vento:
di notte.
Salvatore Quasimodo, *Riposo dell'erba.*

We have already mentioned that there are axiomatizations for theories in $[\mathcal{E}'_n]$ and $[\mathcal{E}_n]$ (for $n \geq 2$) that use only one ternary predicate as primitive notion. This is best possible if the axiomatization is carried out in first order logic. R. M. Robinson [64] has shown than even $\mathcal{UE}'_n$ is not synonymous with any first-order theory whose language has only finitely many binary predicates as primitive notions. By $\mathcal{UE}'_n$ we mean the theory that one obtains by enlarging the language of $\mathcal{E}'_n$ with a binary operation E, whose intended interpretation is $E(ab)$ iff 'the distance between a and b is 1', formally

Definition 2.6.1 *For a geometric theory $\mathcal{T}$ in a first-order language* L*, let* $\mathcal{UT} \overset{\text{def}}{=} Cn(\mathcal{T}$, A2.6.1, A2.6.2, A2.6.3)*, where*

A 2.6.1 $(\exists ab)\,[a \neq b \wedge E(ab)]$,

A 2.6.2 $E(ab) \wedge E(cd) \rightarrow ab \equiv cd$,

A 2.6.3 $E(ab) \wedge ab \equiv cd \rightarrow E(cd)$.

Let $\mathfrak{UC}_2(F) = (F \times F, \mathbf{B}_F, \equiv_F, \mathbf{E}_F)$, with $\mathbf{E}_F(\mathbf{ab})$ iff $\|\mathbf{a} - \mathbf{b}\| = 1$.

That E alone can not serve as primitive notion for a theory in $[\mathcal{UE}'_2]$ can be seen by noting that $\equiv$ is not definable in terms of E in it, since, for any non-Archimedean ordered Euclidean field K, $f : K \times K \rightarrow K \times K$, defined by

$$f(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } \|\mathbf{x}\| < n \text{ for some } n \in \mathbb{N} \\ (x_1, x_2 + 1), & \text{otherwise} \end{cases}$$

preserves $\mathbf{E}_K$ but not $\equiv_K$ (hence we get the desired conclusion by Padoa's method).

We shall however prove that, although not axiomatizable in $\mathrm{L} = \mathrm{L}_E$, plane Euclidean geometry coordinatized by Archimedean ordered Euclidean fields can be axiomatized by using E as the only primitive notion in $\mathrm{L}_{\omega_1\omega}$, the axiomatization being in fact in its constructive fragment $\mathrm{CL}_{\omega_1\omega}$ (cf. [41]).

Let AE be the set of all Archimedean ordered Euclidean ordered fields and let $\mathcal{UE}'_{2;\omega_1\omega} = \bigcap_{F \in AE} Th_{\mathrm{L}'_{\omega_1\omega}} \mathfrak{UC}_2(F)$, where $\mathrm{L}' = \mathrm{L}_{B\equiv}$.

We now want to prove that $\equiv$ is definable from E in $\mathcal{UE}'_{2;\omega_1\omega}$. Let S be the denumerable set of all L-formulas φ with two free variables, for which $\varphi(ab)$ is the sentence stating that 'the distance between a and b is $m/2^n$', where m and n are positive integers. That such formulas exist can be seen from Figures 2.7 and 2.8, where it is shown how to define integer distances and halves respectively. The set $\{m/2^n \mid m, n \in \mathbb{N}\}$ being, in Archimedean ordered

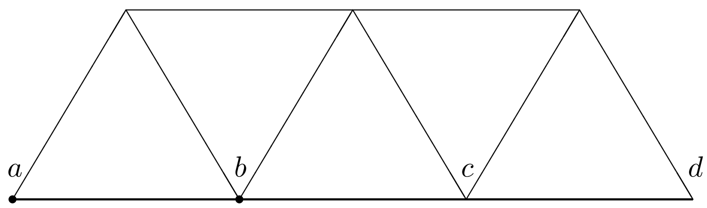

Figure 2.7: If $ab = 1$, then $ac = 2$, $ad = 3$, etc.

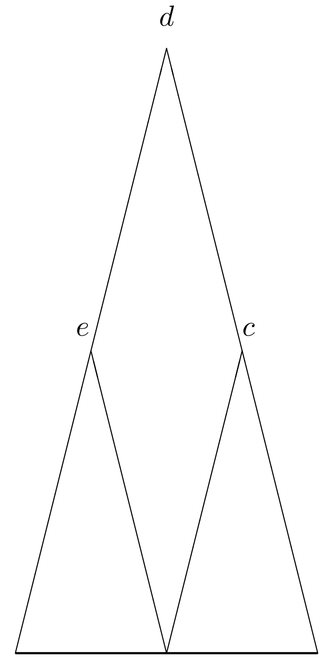

Figure 2.8: If $bf = ef = ed = dc = cb = ca = ea = n$ and $df = db = 2n$, then $ab = n/2$.

fields, dense in the positives, the following is a definition of $\equiv$ in terms of E:

$$ab \equiv cd \quad \leftrightarrow \quad \bigwedge_{\varphi \in S} (\forall x)(\exists y)\,[(\varphi(ax) \wedge \varphi(bx) \rightarrow \varphi(cy) \wedge \varphi(dy)) \wedge (\varphi(cx) \wedge \varphi(dx) \rightarrow \varphi(ay) \wedge \varphi(by))]. \tag{2.23}$$

Since B can be defined in terms of $\equiv$ in $\mathcal{E}_2'$, we can restate, by using (2.23), each axiom in its axiom system Υ as an axiom in $L_{\omega_1\omega}$; the same applies to A2.6.1-A2.6.3. Denoting by Υ' the resulting $L_{\omega_1\omega}$- axiom system and stating the Archimedeanity of the coordinate field by the $L_{\omega_1\omega}$-sentence

A 2.6.4 $(\forall ab)\,[\bigvee_{n=1}^{\infty}((\exists x_1 \ldots x_n)\ E(ax_1) \wedge E(x_nb) \wedge (\bigwedge_{i=1}^{n-1} E(x_ix_{i+1})))]$,

we get the following

Representation Theorem 2.6.1 *$\mathfrak{M} \in Mod(\Upsilon' \cup \{A2.6.4\})$ iff $\mathfrak{M} \simeq \mathfrak{U}_2(F)$, for some Archimedean ordered Euclidean field F, with $\mathfrak{U}_2(F) = (F \times F, \mathbf{E}_F)$.*

whose proof derives from the fact that, since the congruence relation defined by (2.23) will get the $\equiv_F$-interpretation (by the representation theorem for $Cn(\Upsilon) = \mathcal{E}_2'$), E will get the $\mathbf{E}_F$-interpretation (by A2.6.1-A2.6.3). The Archimedeanity of the ordered Euclidean field now follows from A2.6.4.

The axiomatization of Euclidean geometry with only one binary undefined notion (witb 'points' as variables) is best possible, regardless of the language, since there are only two geometric unary predicates: the empty one and the universal one.

There are however some open questions, as to whether the axiomatization with E as single undefined notion is possible in other extensions of first order logic.

A natural candidate would be weak second-order logic $L(II_0)$, since the Archimedeanity of the ordered coordinate field can be expressed in the weak second-order extension of the language L', i. e. in $L'(II_0)$. If possible, such an axiomatization should be preferred to the one in $L_{\omega_1\omega}$, since $L(II_0)$ is weaker than $L_{\omega_1\omega}$ (i. e. the latter has more 'expressive power' (cf. [13])).

A second natural candidate would be $L(Q^2)$, the logic with the Ramsey quantifier Q^2, for the same reason that the Archimedeanity of the ordered coordinate field can be expressed in $L'(Q^2)$ (cf. [14]). $L(Q^2)$ is neither weaker nor stronger than $L_{\omega_1\omega}$; they are incomparable (cf. [13]).

An indication that, in some adequate logic, $\equiv$ is definable from E in the Archimedean ordered case, was read from the Beckman-Quarles theorem (cf. [12]) (simplified statement: 'If $T : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ preserves $\mathbf{E}_{\mathbb{R}}$, then T is an isometry'); the fact that $\mathbb{R}$ may be replaced in this theorem by any Archimedean ordered Euclidean plane has been noted by Farrahi in [23].

An investigation into geometry with a unit length dealing with problems of ruler and compass constructibility was begun in [15] and continued in [94], where it is shown that one can axiomatize plane Euclidean geometry with a unit distance, over Archimedean ordered fields, by universal $CL_{\omega_1\omega}$ sentences in a language with only one binary operation and three individual constants as primitive notions (cf. §4.3.4). The results in [42] and [43], although relevant for geometry with a unit distance, go beyond ruler and compass constructibility. æ

Chapter 3

THE CONSTRUCTIVE SETTING

No existents whatsoever are evident anywhere
that are arisen from themselves, from another,
from both, or from a non-cause.
Nāgārjuna, *Mūlamadhyamakakārikā.*

3.1 Introduction

Tres numerus super omnia.
Decimus Magnus Ausonius, *Griphus tertiarii numeri.*

Although HILBERT's (and to some extent TARSKI's) axiomatization of Euclidean geometry represent an attempt to restate in the framework of modern logic the assumptions made by EUCLID's axioms, there is a central idea about the nature of geometry implicit in the *Elements* that is lost in these modern axiomatizations: the idea that both geometric theorems and proofs are *constructions* (cf. [92], [91], [74]).

The first attempt at a modern axiomatization that captures this essential aspect of 'Greek' geometry was a paper by N. MOLER and P. SUPPES in 1968 ([46]), who gave a quantifier-free axiom system for plane Euclidean geometry over Pythagorean ordered fields in a first-order language with variables for 'points' and with three individual constants a_0, a_1, a_2 and two quaternary operation symbols S and I as primitive notions. The primitive notions a_0, a_1, a_2, S and I have the following intuitive meanings:

a_0, a_1, a_2 are three non-collinear points,

$S(xyuv) = w$ iff the point w is as distant from u on the ray $\overrightarrow{uv}$ as y is from x,
provided that $u \neq v \vee (u = v \wedge x = y)$,
arbitrary, otherwise,

$I(xyuv) = w$ iff w is the point of intersection of the lines xy and uv,
provided that $x \neq y \wedge u \neq v \wedge \neg(L(xyu) \wedge L(xyv)) \wedge xy \not\parallel uv$,
arbitrary, otherwise,

Some minor inconsistencies in [46], noticed in [71], were corrected by H. Seeland [73], who also gave a quantifier-free axiom system for plane Euclidean geometry over Euclidean fields, in a language enlarged with a third quaternary symbol C, having the intuitive meaning

$$C(xyuv) = w \quad \text{iff} \quad \begin{array}{l} w \text{ is the point of intersection of the circle centered at } x \text{ and} \\ \text{passing through } y \text{ with the segment } uv, \\ \text{provided that } x \neq y,\ u \text{ lies inside and } v \text{ lies outside the circle,} \\ \text{arbitrary, otherwise,} \end{array}$$

We shall prove in the paragraphs that follow, that one can axiomatize a wider class of plane Euclidean geometries — namely rectangular planes, metric-Euclidean planes, Euclidean planes, Cartesian planes over ordered, Pythagorean ordered as well as over Euclidean ordered fields — by using three individual constants and only *ternary* operations as primitive notions. These quantifier-free axiomatizations will also turn out to be most simple, regardless of language (see Simplicity Criterion 1.2.4) and (up to some conjectures) constructively expressed in the simplest possible language.

These axiomatizations will also shed new light on the very nature of these geometries and on the relationship between these geometries.

No finite set of binary operations and individual constants can be used to axiomatize (even allowing the use of the existential quantifier) a theory in $[\mathcal{E}_2']$ or in $[\mathcal{E}_2']'$ and the same applies to all other plane geometries studied (cf. [22, p. 77-80], [47]), so a quantifier-free axiomatization using only ternary operations is, with respect to the maximal arity of the operation symbols involved, best possible.

3.2 Metric-Euclidean planes

Je vous conseille de douter de tout, excepté que
les trois angles d'un triangle sont égaux à deux droits.
Voltaire.

The Euclideanity of a Euclidean plane (or equivalently of a Gaußian plane) may be considered as being determined by its affine structure (i. e. by the fact that an Euclidean plane is an affine plane), or as being determined by its Euclidean metric. Taking the second approach, one may ask what the most general 'planes' with a Euclidean metric are, and whether having a Euclidean metric implies the affine structure (i. e. the intersection of non parallel lines). It was shown by M. Dehn [17] as early as 1900 that this is not the case, i. e. that there are planes with a Euclidean metric (to be precise 'metric-Euclidean planes') that are not Euclidean (i. e. where the parallel axiom does not hold).

We shall first provide quantifier-free axiomatizations for such 'Euclidean-like' classes of structures, in which the Euclidean parallel axiom need not hold, called metric-Euclidean planes and rectangular planes of characteristic $\neq 2$.

Metric-Euclidean planes were introduced by F. Bachmann in [10], [5], [6], [7], as a plane geometry with Euclidean metric, without order, where the Euclidean parallel axiom need not hold.

A first example of a rectangular plane is mentioned in [5, §4]. Rectangular planes were introduced in [38] by a mixed geometric-group-theoretic axiom system, as a generalization of metric-Euclidean planes, where perpendicular lines need not intersect. A purely geometric axiomatization for rectangular planes (*Rechtseitebenen*) of characteristic $\neq 2$, in a language without operation symbols, was provided by R. Stanik [78], where further references on planes with a Euclidean metric can be found.

The theory of metric planes is that common substratum of Euclidean and non-Euclidean plane geometries that can be expressed in terms of incidence and orthogonality, where order, free mobility, and the intersection of non-orthogonal lines is ignored.

Axiom systems both for the motion group of a metric plane and for metric planes themselves were given in [7]. The metric of a metric plane is called Euclidean if there exists a rectangle in that plane. An axiom system in the language $\mathrm{L} = \mathrm{L}_{L\equiv}$, with individual variables to be interpreted as 'points' and where L is a ternary predicate standing for collinearity and $\equiv$ a quaternary predicate standing for equidistance (i.e. $L(abc)$ should read 'a, b, c are three collinear points' and $ab \equiv cd$ should be read 'the segment ab is congruent to the segment cd'), can be extracted from [77]. Consider the following axioms:

A 3.2.1 $L(aba)$,

A 3.2.2 $L(abc) \to L(cba) \wedge L(bac)$,

A 3.2.3 $a \neq b \wedge L(abc) \wedge L(abd) \to L(acd)$,

A 3.2.4 $ab \equiv ab$,

A 3.2.5 $aa \equiv bb$,

A 3.2.6 $(\forall abca'b')(\exists^{=1}c')\,[a \neq b \wedge L(abc) \wedge ab \equiv a'b' \to L(a'b'c') \wedge ac \equiv a'c' \wedge bc \equiv b'c']$,

A 3.2.7 $\neg L(abx) \wedge L(abc) \wedge L(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c'$
$\wedge\, ax \equiv a'x' \wedge bx \equiv b'x' \to xc \equiv x'c'$,

A 3.2.8 $(\forall abx)(\exists^{=1}x')\,[\neg L(abx) \to x' \neq x \wedge ax \equiv ax' \wedge bx \equiv bx']$,

A 3.2.9 $\neg L(abx) \wedge \neg L(aby) \wedge ax \equiv ax' \wedge bx \equiv bx' \wedge x \neq x' \wedge ay \equiv ay' \wedge by \equiv by'$
$\wedge\, y \neq y' \to xy \equiv x'y'$,

A 3.2.10 $(\forall abxx')(\exists y)\,[\neg L(abx) \wedge x' \neq x \wedge ax \equiv ax' \wedge bx \equiv bx' \to L(aby) \wedge L(xx'y)]$,

A 3.2.11 $(\forall ab)(\exists^{=1}b')\,[a \neq b \to L(abb') \wedge ab \equiv ab' \wedge b' \neq b]$,

A 3.2.12 $(\forall xyzab)(\exists c)\,[x \neq y \wedge y \neq z \wedge z \neq x \wedge L(xyz) \wedge L(xya) \wedge ax \equiv ay \wedge L(xyb)$
$\wedge\, by \equiv bz \to cz \equiv cx]$,

A 3.2.13 $\neg L(xyz) \wedge L(axy) \wedge ax \equiv ay \wedge L(bzy) \wedge bz \equiv by \wedge L(cxz) \wedge cx \equiv cz \to \neg L(abc)$,

A 3.2.14 $(\exists a_1a_2a_3a_4a'_1a'_2a'_3a'_4)\,[(\bigwedge_{i\neq j} a_i \neq a_j) \wedge (\bigwedge_{i=1}^{4} a_i \neq a_{i'} \wedge L(a_ia_{i+1}a'_{i+1}) \wedge a_ia_{i+1} \equiv a_ia'_{i+1} \wedge a_{i-1}a_{i+1} \equiv a_{i-1}a'_{i+1})]$,[1]

A 3.2.15 $(\exists a_1a_2a_3a_4a'_1a'_2a'_3a'_4)\,[(\bigwedge_{i\neq j} a_i \neq a_j) \wedge (\bigwedge_{i=1}^{4} a_i \neq a_{i'} \wedge L(a_ia_{i+1}a'_{i+1}) \wedge a_ia_{i+1} \equiv a_ia'_{i+1} \wedge a_{i-1}a_{i+1} \equiv a_{i-1}a'_{i+1}) \wedge a_1a_2 \equiv a_2a_3]$.

Here are, in words, the statements some of the axioms make:

A3.2.6: 'for lines that contain congruent segments we can transport in a certain rigid and unique way further segments from one line to the other'; A3.2.7 is closely related to TARSKI's 'five-segment axiom' A1.1.5; A3.2.8: 'the reflection of a point in a line exists and is unique'; A3.2.9: 'reflections are rigid motions (i. e. isometries)'; A3.2.10: 'orthogonal lines intersect'; A3.2.11: 'the reflection of a point in a point exists and is unique'; A3.2.12: 'if two pairs of three collinear points have midpoints than the third one has a midpoint as well (which will be the intersection of the perpendicular from c to the line on which x, y, z lie)'; A3.2.13: the midpoints of the sides of a triangle are not collinear; A3.2.14: 'there is a rectangle (with vertices a_1, a_2, a_3, a_4)'; A3.2.15: 'there is a square (with vertices a_1, a_2, a_3, a_4)'. The axiom system in [77] has axiom

A 3.2.16 $(\forall aba'b')(\exists uvu'v'z)\,[\neg L(aba') \wedge a' \neq b' \rightarrow \neg L(abz) \wedge \neg L(a'b'z) \wedge L(abu) \wedge L(a'b'u') \wedge L(uzu') \wedge L(abv) \wedge L(a'b'v') \wedge L(vzv') \wedge u \neq v]$

instead of A3.2.9. Since A3.2.16 was used only to prove A3.2.9, we can prove all the theorems needed for the representation theorem with our modified axiom system. We chose to replace A3.2.16 by A3.2.9, since we were unable to prove A3.2.16 in our constructive axiom system, although it is a consequence of those axioms.

Let $\Sigma = \{$A3.2.1-A3.2.14, A2.1.1, A2.1.2, A1.1.1, A1.1.3$\}$, $\Sigma' = \{$A3.2.1-A3.2.13, A3.2.15, A2.1.1, A2.1.2, A1.1.1, A1.1.3$\}$, $\mathcal{ME} \stackrel{\text{def}}{=} Cn(\Sigma)$, $\mathcal{ME}' \stackrel{\text{def}}{=} Cn(\Sigma')$.

K. SÖRENSEN [77] proved that the models of $\{$A3.2.1-A3.2.13, A2.1.1, A2.1.2, A1.1.1, A1.1.3$\}$ are metric planes, so the models of Σ are metric-Euclidean planes, and the models of Σ' are metric-Euclidean planes with bisectable right angles. Conversely, it is easy to check that the axioms in $\{$A3.2.1-A3.2.13, A2.1.1, A2.1.2, A1.1.1, A1.1.3$\}$ are valid in all metric-Euclidean planes (as defined in [10, §19]). To see this, notice that any metric-Euclidean plane $\mathfrak{ME}$ can be embedded in some Euclidean plane $\mathfrak{E} = \mathfrak{D}_2(F,k)$ (cf. §2.1 for a definition of Euclidean planes) such that:

(Z) 'On any line g in $\mathfrak{E}$ that passes through a point P in $\mathfrak{ME}$, there is a second point $\mathsf{Q} \neq \mathsf{P}$ that is in $\mathfrak{ME}$ as well',

and hence is isomorphic to a certain structure that can be described algebraically (cf. [7]), so checking the axioms becomes a matter of simple algebraic computations.

According to the algebraic characterization of metric-Euclidean planes from [7] (cf. also [36, III. 6]) we have

Representation Theorem 3.2.1 *$\mathfrak{M} \in Mod(\mathcal{ME})$ iff $\mathfrak{M} \simeq \langle E, \mathbf{L}_K, \equiv_{(K,k)}\rangle$, where K is a field of characteristic $\neq 2$, $k \in K$, $-k \notin K^2$, $L = K(\sqrt{-k})$, $\|z\| = x^2+ky^2$ for $z = x+y\sqrt{-k}$,*

[1] Addition in the indices is *mod 4* both in A3.2.14 and in A3.2.15.

$L_1 = \{z \in L \mid \|z\| = 1\}$, *$R = R(K,k)$ the subring of L generated by L_1, $E \subset L$ is an R-module with* $0, 1 \in E$, *satisfying* $(\forall s \in L)(\forall x \in E)\|s\| = 1 \Rightarrow \frac{1}{2}(x + sx) \in E$,
$\mathbf{L}_K(\mathbf{xyu})$ *iff* $K(\mathbf{x}-\mathbf{u}) = K(\mathbf{y}-\mathbf{u})$ *or* $\mathbf{x} = \mathbf{u}$ *or* $\mathbf{y} = \mathbf{u}$,
$\mathbf{xy} \equiv_{(K,k)} \mathbf{uv}$ *iff* $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{u}-\mathbf{v}\|$, *with* $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in E$.
$\mathfrak{M} \in Mod(\mathcal{ME}')$ *iff* $\mathfrak{M} \simeq \langle E, \mathbf{L}_K, \equiv_{(K,1)}\rangle$, *with notations as above, with* $k = 1$.

In other words, the point-set of a metric-Euclidean plane of characteristic $\neq 2$ (i. e. the universe of a model of Σ) is isomorphic to a subset E of $L = K(\sqrt{-k})$, which satisfies
(i) $(E, +)$ is a subgroup of $(L, +)$ and $1 \in E$,
(ii) $(\forall s \in L)\|s\| = 1 \Rightarrow s \cdot E \subseteq E$,
(iii) $(\forall s \in L)(\forall x \in E)\|s\| = 1 \Rightarrow \frac{1}{2}(x + sx) \in E$.
It inherits the collinearity and congruence relations from the Euclidean plane over (K, k) (or, equivalently of the Gaußian plane over (L, K)) (cf. [54]).

This means, geometrically speaking, that this point-set contains 0, 1 and is closed under translations and rotations around 0, and contains the midpoints of any point-pair consisting of an arbitrary point and its image under a rotation around 0.

Note that the underlying set of $R(K, k)$ is the additive group generated by L_1.

3.3 A constructive axiom system for metric-Euclidean planes

We shall provide a quantifier-free axiom system for metric-Euclidean planes in a language $\mathrm{L}_{me} = \mathrm{L}(a_0, a_1, a_2, P, F)$, where a_0, a_1, a_2 are individual constants (standing for three non-collinear points), P and F ternary operations, to be read as $P(abc) = d$ iff '$abdc$ is a parallelogram, i.e. $ab \| cd \wedge ac \| bd$' and $F(abc) = d$ iff 'd is the footpoint of the perpendicular from c to the line ab (if $a \neq b$; a itself in the degenerate case $a = b$)'.

The choice of the language is an arguably natural one, both because the operations P and F are needed for a natural definition of the congruence relation, and because their central importance was already noticed by F. Bachmann, who singled them out in the following

Theorem 3.3.1 *([7, p. 275]) If one transports to a point* O *the vectors that one can obtain in a Euclidean plane starting from two orthogonal vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *by a finite sequence of the operations*

1) Given the vectors $\mathbf{a}$, $\mathbf{b}$, *form the difference vector* $\mathbf{a} - \mathbf{b}$,

2) Given the vectors $\mathbf{a}$, $\mathbf{b}$ *with* $\mathbf{b} \neq 0$, *form the orthogonal projection of* $\mathbf{a}$ *into* $\mathbf{b}$,

then the endpoints of these transported vectors, together with the collinearity and congruence relations induced from the underlying Euclidean plane, form a metric-Euclidean plane satisfying (Z).

In order to formulate the axioms in a more readable way, we shall use the following abbreviations:

$$\sigma(ba) \stackrel{\text{def}}{=} P(abb) \tag{3.1}$$

$$R(abc) \stackrel{\text{def}}{=} \sigma(F(abc)c) \tag{3.2}$$

$$L(abc) \stackrel{\text{def}}{\leftrightarrow} F(abc) = c \vee a = b \tag{3.3}$$

$$I(abc) \overset{\text{def}}{\leftrightarrow} \sigma(F(cba)b) = c \tag{3.4}$$

$$ab \equiv cd \overset{\text{def}}{\leftrightarrow} I(cP(abc)d). \tag{3.5}$$

These may be read as '$\sigma(ba)$ is the point obtained by reflecting a in b'; '$R(abc)$ is the point obtained by reflecting c in ab' (hence coincides with $\sigma(ca)$ if $a = b$); $L(abc)$ iff 'a, b, c are collinear'; $I(abc)$ iff 'ab is congruent to ac' and $ab \equiv cd$ iff 'ab is congruent to cd'.

Consider the following axioms:

A 3.3.1 $P(abc) = P(acb)$,

A 3.3.2 $P(abc) = c \to a = b$,

A 3.3.3 $\sigma(ax) = \sigma(bx) \to a = b$,

A 3.3.4 $a \neq b \wedge c \neq d \wedge F(abc) = c \wedge F(abd) = d \to F(abx) = F(cdx)$,

A 3.3.5 $\neg L(abx) \wedge x \neq x' \wedge I(axx') \wedge I(bxx') \to x' = R(abx)$,

A 3.3.6 $L(ab\sigma(ab))$,

A 3.3.7 $a \neq b \to xy \equiv R(abx)R(aby)$,

A 3.3.8 $P(abd) = P(cP(abc)d)$,

A 3.3.9 $I(oab) \wedge I(obc) \to I(oac)$,

A 3.3.10 $I(oab) \to I(o'P(oao')P(obo'))$,

A 3.3.11 $\neg L(a_0a_1a_2)$,

A 3.3.12 $a_0 \neq a_1 \wedge a_1 \neq a_2 \wedge a_2 \neq a_0 \wedge F(a_0a_1a_2) = a_0 \wedge I(a_0a_1a_2)$.

Notice that A3.3.8 is the minor Desargues axiom and that A3.3.9 is what Bachmann ([10, §4,1]) calls a *Mittelsenkrechtensatz*, for it says that if in a triangle two of the perpendicular bisectors meet, then the third one is concurrent with the first two.

Let $\mathcal{CME} = Cn_{\mathrm{L}_{me}}$(A3.2.1, A3.3.1-A3.3.11) and $\mathcal{CME}' = Cn_{\mathrm{L}_{me}}$(A3.2.1, A3.3.1-A3.3.10, A3.3.12), i. e. both $\mathcal{CME}$and $\mathcal{CME}'$ are L_{me}-theories, as we considered the axioms A3.2.1, A3.3.1-A3.3.12 to be L_{me}-axioms. Let $\Delta = \{$A3.2.1, A3.3.1-A3.3.11, (3.1), (3.2), (3.3), (3.4), (3.5)$\}$. In order to show that models of $\mathcal{CME}$ are metric-Euclidean planes, we shall prove that $\Delta \vdash \Sigma$. In the derivations that follow 'true' will mean 'a consequence of Δ' (i. e. in $Cn(\Delta)$) and all numbered formulas are true. Note that the only axiom in the axiom system for $\mathcal{CME}$, that tells us that the metric is Euclidean is A3.3.8, the minor Desargues axiom. All other axioms are true in absolute planes, i. e. in $\mathcal{A}_2$ (as defined in §2.5), with $P(abc)$ defined as having the effect of $\sigma(M(bc)a)$. More generally, all other axioms are true in metric planes in which every segment has a midpoint, with P defined as above.

Suppose $P(abx) = P(aby)$. We want to deduce that $x = y$. By A3.3.8 and A3.3.1 we have $P(abx) = P(yxP(aby))$, hence $P(yxP(aby)) = P(aby)$. Therefore, by A3.3.2,

$$P(abx) = P(aby) \to x = y. \tag{3.6}$$

By A3.3.8 we have $P(abb) = P(aP(aba)b)$, so using A3.3.1 and (3.6) we deduce

$$P(aba) = P(aab) = b. \tag{3.7}$$

Notice that, by (3.7) and (3.5), $ab \equiv ac \leftrightarrow I(abc)$ is true, which is why we gave the same reading to $ab \equiv ac$ and $I(abc)$, and why we shall use them interchangeably without further notice. With $c = b$, $d = a$ A3.3.8 becomes $P(bP(abb)a) = P(aba)$, which, by (3.7), A3.3.1, (3.1), implies

$$P(ba\sigma(ba)) = b. \tag{3.8}$$

Let $a' = \sigma(ba)$. By A3.3.8 we have $P(a'ba') = P(bP(a'bb)a')$, i. e. $P(bP(a'bb)a') = b$ (by (3.7)). By (3.8) this means that $P(bP(a'bb)a') = P(baa')$. Using A3.3.1 and (3.6) we conclude that $P(a'bb) = a$, i. e. by (3.1)

$$\sigma(b\sigma(ba)) = a \tag{3.9}$$

We also have

$$\sigma(aa) = a \ \text{(by (3.1), (3.7))}. \tag{3.10}$$

$$\sigma(ba) = b \to a = b \text{ (by A3.3.2), and } \sigma(ba) = a \to a = b \text{ (by A3.3.3, (3.10))}. \tag{3.11}$$

We deduce from (3.2), (3.11), (3.3), (3.10) that

$$a \neq b \wedge L(abc) \to R(abc) = c \text{ and } R(abc) = c \to L(abc). \tag{3.12}$$

We now turn to the proof of A3.2.4. Let $x \neq y$. By A3.3.7 we get $xy \equiv R(y\sigma(yx)x)R(y\sigma(yx)y)$, since $y \neq \sigma(yx)$ (by (3.11)). By (3.9) $\sigma(y\sigma(yx)) = x$, hence, $L(y\sigma(yx)x)$ (by A3.3.6), hence $R(y\sigma(yx)x) = x$ (by (3.12) and the fact that, according to (3.11), $y \neq \sigma(yx)$). By A3.2.1, (3.12) and $y \neq \sigma(yx)$, we get $R(y\sigma(yx)y) = y$, hence $xy \equiv xy$. Since, by A3.3.11, $a_0 \neq a_1$, we must have, for any x, $x \neq a_0 \vee x \neq a_1$. We may assume w. l. o. g. that $x \neq a_0$. By A3.3.7 we get $xx \equiv R(a_0\sigma(a_0x)x)R(a_0\sigma(a_0x)x))$. Using the same reasoning as above, we get $R(a_0\sigma(a_0x)x) = x$, hence $xx \equiv xx$, so we have proved A3.2.4.

The statement A3.2.4 makes is equivalent with $I(abb)$, i. e. $\sigma(F(bba)b) = b$ (by (3.4)), whence, using (3.11), we get

$$F(bba) = b. \tag{3.13}$$

With $c = a$ and $d = \sigma(ab)$ A3.3.4 becomes

$$a \neq b \wedge a \neq \sigma(ab) \wedge F(aba) = a \wedge F(ab\sigma(ab)) = \sigma(ab) \to F(abx) = F(a\sigma(ab)x).$$

For $a \neq b$ the antecedent of this implication is true — as $a \neq \sigma(ab)$ is implied by (3.11), $F(aba) = a$ by A3.2.1, and $F(ab\sigma(ab)) = \sigma(ab)$ by A3.3.6 — so the consequent must be true, i.e.

$$a \neq b \to F(abx) = F(a\sigma(ab)x). \tag{3.14}$$

$$a \neq b \to F(a\sigma(ab)\sigma(a\sigma(ab))) = \sigma(a\sigma(ab)) \ \text{(by A3.3.6, (3.11))} \tag{3.15}$$

Since $\sigma(a\sigma(ab)) = b$ (by (3.9)), (3.15) becomes, for $a \neq b$ (for $a = b$ it follows from (3.10), (3.13)))

$$F(a\sigma(ab)b) = b. \tag{3.16}$$

(3.16) and (3.14) (with $x = b$) imply (using (3.3) as well)

$$L(abb). \tag{3.17}$$

$$F(abx) = F(bax) \text{ (by A3.2.1, (3.17), A3.3.4).} \tag{3.18}$$

$$R(aab) = \sigma(ab) \text{ (by (3.2), (3.13)).} \tag{3.19}$$

$$R(abx) = R(bax) \text{ (by (3.2), (3.18)).} \tag{3.20}$$

$$R(abb) = b \text{ (by (3.17), (3.12) (for } a \neq b\text{) and (3.19), (3.10) (for } a = b\text{)).} \tag{3.21}$$

Using A3.2.4 (i. e. $I(xyy)$) and (3.5), we get

$$ab \equiv cP(abc). \tag{3.22}$$

$$a \neq b \wedge c \neq d \wedge L(abc) \wedge L(abd) \to (L(abx) \to L(cdx)) \text{ (by A3.3.4, (3.3)).} \tag{3.23}$$

Letting $c = b$ and $d = a$, $a \neq b$, we get a true antecendent in (3.23), hence a true consequent, i. e.

$$a \neq b \to (L(abx) \to L(bax)). \tag{3.24}$$

Letting $d = b$ and $x = a$, (3.23) becomes (by (3.17), A3.2.1)

$$a \neq b \wedge c \neq b \wedge L(abc) \to L(cba) \tag{3.25}$$

To prove A3.2.2, notice that in both (3.24) and (3.25) the consequents are true regardless of the antecendents (in (3.24), if $a = b$, then the consequent is a tautology; if $a = b$, then (3.25) is a consequence of $L(aac) \to L(caa)$ (in which both antecedent and consequent are true (by (3.3) and (3.17) respectively); if $c = d$ then (3.25) is a consequence of $L(abb) \to L(bba)$ (in which both antecedent and consequent are true)). This proves that A3.2.2 is true.

With $x = a$, (3.23) becomes (by A3.2.1) $a \neq b \wedge c \neq d \wedge L(abc) \wedge L(abd) \to L(cda)$, which, using A3.2.2, implies A3.2.3 with $c \neq d$ added to its antecendent. If $c = d$, then A3.2.3 follows from (3.3), hence A3.2.3 is true.

Let $u \neq v$ and $F(uvy) \neq y$. $F(uvy)$ must be different from either u or v. We can assume w. l. o. g. (by (3.18)) that $F(uvy) \neq u$. We deduce $I(uyR(uvy))$ (by A3.3.7, (3.21), (3.20)), i. e. $\sigma(F(R(uvy)yu)y) = R(uvy)$ (by (3.4)). On the other hand $\sigma(F(uvy)y) = R(uvy)$ (by (3.2)), hence $F(R(uvy)yu) = F(uvy)$ (by A3.3.3). Together with our assumptions on u, v, y, this implies that the antecedent of A3.3.4 becomes true for $a = F(uvy)$, $b = y$, $c = R(uvy)$, $d = y$, $x = u$ (by A3.3.6, (3.17), and the assumption that $a \neq b$, which in turn implies that $c \neq d$ as well (by (3.12)), hence so does the consequent, i. e. $F(F(uvy)yu) = F(R(uvy)yu)$, hence

$$F(F(uvy)yu) = F(uvy). \tag{3.26}$$

Although (3.26) was proved only for $u \neq v$, $F(uvy) \neq y$ and $F(uvy) \neq u$, it is true for all u, v, y (by (3.13), A3.2.1 for $u = v$, by (3.13) for $F(uvy) = y$ and by A3.2.1 for $F(uvy) = u$).

Suppose $ab \equiv cd$ and $cd \equiv ef$, i. e. $I(cP(abc)d)$ and $I(eP(cde)f)$ (by (3.5)). We want to conclude that $ab \equiv ef$. $I(cP(abc)d) \to I(eP(cP(abc)e)P(cde))$ (by A3.3.10), $I(eP(cP(abc)e)P(cde)) \wedge I(eP(cde)f) \to I(eP(cP(abc)e)f)$ (by A3.3.9), $P(cP(abc)e) = P(abe)$ (by A3.3.8), and hence $I(eP(abe)f)$, i. e. $ab \equiv ef$ (by (3.5)), so A2.1.2 is true.

$$P(cP(abc)a) = P(aba) = b \text{ (by A3.3.8, (3.7))}. \tag{3.27}$$

Suppose $I(abc)$, i. e. $\sigma(F(bca)b) = c$ (by (3.4), (3.18)), so $\sigma(F(bca)c) = \sigma(F(bca)\sigma(F(bca)b))$ and, since $\sigma(F(bca)\sigma(F(bca)b)) = b$ (by (3.9)), we get $I(acb)$ (by (3.4)). This proves that

$$I(abc) \to I(acb). \tag{3.28}$$

Suppose $ab \equiv cd$, i. e. $I(cP(abc)d)$ (by (3.5)). We want to prove that $cd \equiv ab$. By A3.3.10, we deduce $I(aP(cP(abc)a)P(cda))$, hence $I(abP(cda))$ (by (3.27)), and from (3.28) we deduce $I(aP(cda)b)$, i. e. $cd \equiv ab$ (by (3.5)), so A2.1.1 is true.

$I(bP(aab)b)$ is true (by (3.7), (3.13), (3.10)), i.e. A3.2.5 is true (by (3.5)).

To prove A1.1.1, let $a \neq b$. Then $I(bP(abb)a) \leftrightarrow I(b\sigma(ba)a) \leftrightarrow \sigma(F(a\sigma(ba)b)\sigma(ba)) = a$ (by (3.1), (3.4)) and, since $F(a\sigma(ba)b) = b$ (by A3.3.6, A3.2.2), we deduce that $I(bP(abb)a) \leftrightarrow \sigma(b\sigma(ba)) = a$; hence $I(bP(abb)a)$ is true (by (3.9)) and so is A1.1.1 for $a \neq b$ (by (3.5)). For $a = b$ A1.1.1 follows from A3.2.5, and hence A1.1.1 is true for all a, b.

To prove A1.1.3, let $ab \equiv cc$. Then, $\sigma(F(cP(abc)c)P(abc)) = c$ (by (3.4), (3.5)) and, since $F(cP(abc)c) = c$ (by A3.2.1 and (3.13)), we get $\sigma(cP(abc)) = c$, i. e. $P(P(abc)cc) = c$ (by (3.1)), hence $a = b$ (by applying A3.3.2 twice), so A1.1.3 is true.

$$\sigma(ba) = P(cP(abc)b) \text{ (by A3.3.8, (3.1))}. \tag{3.29}$$

We shall prove that A3.3.7 is true even if $a = b$, i. e. that $xy \equiv \sigma(ax)\sigma(ay)$ (by (3.19)). Since $xy \equiv aP(xay)$ (by (3.22), A3.3.1), $aP(xay) \equiv \sigma(ax)P(P(xay)\sigma(ax)a)$ (by (3.22), A3.3.1, A2.1.2, A1.1.1), we get $xy \equiv \sigma(ax)P(P(xay)\sigma(ax)a)$ (by A2.1.2). Since $\sigma(ax) = P(yaP(xay))$ (by (3.29), A3.3.1 and $P(P(xay)P(yaP(xay))a) = \sigma(ay)$ (by (3.29)), we get that $P(P(xay)\sigma(ax)a) = \sigma(ay)$, hence $xy \equiv \sigma(ax)\sigma(ay)$, i. e.

$$xy \equiv R(abx)R(aby). \tag{3.30}$$

$$a \neq b \wedge L(abx) \wedge I(xab) \to b = \sigma(xa) \text{ (by (3.4), (3.18), (3.3))}. \tag{3.31}$$

From (3.31) and A3.3.3 we get

$$a \neq b \wedge L(abx) \wedge L(abx') \wedge I(xab) \wedge I(x'ab) \to x = x'. \tag{3.32}$$

Let us now prove

$$R(abR(abx)) = x. \tag{3.33}$$

In order to prove (3.33), we first show that

$$\neg L(abx) \to \neg L(abR(abx)). \tag{3.34}$$

Suppose $a \neq b \wedge L(abR(abx))$. Then, by A3.3.7, we get $xR(abx) \equiv R(abx)R(abR(abx))$, i. e. , since $R(abR(abx)) = R(abx)$ (by (3.12)), $xR(abx) \equiv R(abx)R(abx)$, from which we conclude that $x = R(abx)$ (by A1.1.3), i. e. $L(abx)$. This proves (3.34).

Suppose now $\neg L(abx)$ and let $x' = R(abx)$. We have $\neg L(abx')$ (by (3.34)), $I(ax'x)$, $I(bx'x)$ (by (3.36), A2.1.1) and $x \neq x'$ (by (3.12)), so we can apply A3.3.5 to conclude $x = R(abx')$, which is (3.33), if $\neg L(abx)$. If $a \neq b \wedge L(abx)$ then (3.33) is an easy consequence of (3.12) and if $a = b$ (3.33) is equivalent to (3.9) (by (3.19)). This proves (3.33).

Also note that, by A3.3.7 and A1.1.3

$$a \neq b \wedge R(abx) = R(aby) \to x = y. \tag{3.35}$$

From (3.30), (3.12), A3.2.1 we deduce

$$I(uyR(uvy)), \text{ whence (by (3.19))}\, I(uy\sigma(uy)), \tag{3.36}$$

$$I(abc) \to R(aF(cba)b) = c \text{ (by (3.4), (3.18), (3.26), (3.2))}. \tag{3.37}$$

Let now a, b, c be different and such that $L(abc)$. We want to prove that $L(aR(axb)R(axc))$. For a, b, c, x such that $L(axb)$, this is obvious as then $L(axc)$ as well (by A3.2.2, A3.2.3) and hence $R(axb) = b$ and $R(axc) = c$ (by (3.12)). Suppose $\neg L(axb)$ and $\neg L(aR(axb)R(axc))$. Let $r = R(aR(axb)R(axc))$. Then $r \neq R(axc)$ (by (3.12)), hence $R(axr) \neq R(axR(axc))$ (by (3.35)), i. e. , since $R(axR(axc)) = c$ (by (3.33)), $R(axr) \neq c$. Using (3.36) three times, as well as A2.1.2, we get $I(acR(axr))$, and, using (3.30) twice, (3.36), (3.35) and A2.1.2, $I(bcR(axr))$. Since $r \neq R(axc)$, we also have $c \neq R(axr)$ (by (3.35), (3.33)), so we can infer, using A3.3.5, that $R(axr) = R(abc)$. Since $L(abc)$, $R(abc) = c$ (by (3.12)), we have $R(axr) = c$, a contradiction. Note that

$$L(abc) \to L(aR(axb)R(axc)) \tag{3.38}$$

is also valid if a, b, c are not different.

$P(bP(aa'b)c) = P(aa'c)$ (by A3.3.8), and, since $bc \equiv P(aa'b)P(bP(aa'b)c)$ (by (3.22), A3.3.1), we deduce

$$bc \equiv P(aa'b)P(aa'c). \tag{3.39}$$

Let $a \neq b$, $L(abc)$, $b' = P(aba')$ and $c' = P(aca')$. In order to prove that $L(a'b'c')$, we assume its falsehood. Then $R(a'b'c') \neq c'$ (by (3.12)), and, for $x = P(a'R(a'b'c')a)$, we have $x \neq c$ (by A3.3.1, (3.6) and $P(a'c'a) = c$ (by (3.27)). We also obtain $ac \equiv a'c'$ (by (3.22)), $a'c' \equiv a'R(a'b'c')$ (by (3.36), $a'R(a'b'c') \equiv ax$ (by (3.22)), hence, by A2.1.2, $I(acx)$. We also have $bc \equiv b'c'$ (by (3.39) and A3.3.1), $b'c' \equiv b'R(a'b'c')$ (by (3.36)), and $b'R(a'b'c') \equiv bx$ (since $P(a'ab') = b$ (by (3.27), A3.3.1), hence $P(b'bR(a'b'c')) = P(a'aR(a'b'c'))$ (by A3.3.8), i. e. $P(b'bR(a'b'c')) = x$ (by A3.3.1), therefore $b'R(a'b'c') \equiv bx$ (by (3.22), A3.3.1)), hence $I(bcx)$ (by A2.1.2). We have thus proved that, assuming $\neg L(a'b'c')$, we get $x \neq c \wedge I(acx) \wedge I(bcx)$. If $L(xca)$, then we also have $L(xcb)$ (by A3.2.2, A3.2.3), hence we have $x \neq c \wedge L(xca) \wedge L(xcb) \wedge I(axc) \wedge I(bxc)$ (by A2.1.2, A2.1.1, A1.1.1), i. e., according to (3.32), $a = b$, a contradiction. If $\neg L(xca)$, then $\neg L(abx)$ as well (by A3.2.2, A3.2.3), and we have $\neg L(abx) \wedge x \neq c \wedge I(axc) \wedge I(bxc)$ (by A1.1.1), so, applying A3.3.5, we get $c = R(abx)$, i. e. $R(abc) = x$ (by (3.33)), i. e. $c = x$ (by (3.12)), a contradiction. So we have proved, for $a \neq b$,

$$L(abc) \to L(a'P(aba')P(aca')), \tag{3.40}$$

which is true for $a = b$ as well (by (3.7), (3.3)).

We now turn to the proof of A3.2.6. Let $b'' = P(aba')$ and $c'' = P(aca')$. For the existence statement in A3.2.6, we claim that $c' = R(a'F(b'b''a')c'')$ satisfies all the requirements. $ac \equiv a'c''$ (by (3.22)), $bc \equiv b''c''$ (by (3.39), A3.3.1) and $L(a'b''c'')$ (by (3.40)), hence $L(R(a'F(b'b''a')a')R(a'F(b'b''a')b'')c')$ (by (3.38)), i. e. , since $R(a'F(b'b''a')a') = a'$ (by A3.2.1, (3.12), (3.21)) and $R(a'F(b'b''a')b'') = b'$ (by (3.37)), $L(a'b'c')$. We also have $a'c'' \equiv a'c'$ (by (3.36)), $b''c'' \equiv b'c'$ (by (3.30)), hence $ac \equiv a'c'$ and $bc \equiv b'c'$ (by A2.1.2). The uniqueness of c' follows from the fact that a second point x that satisfies all the conditions of c' would have to satisfy $L(c'xa')$, $L(c'xb')$ (by A3.2.2, A3.2.3), as well as $I(a'c'x)$ and $I(b'c'x)$ (by A2.1.2, A2.1.1), and therefore, if $c' \neq x$, we should have $a' = b'$ (by (3.32)), which would in turn imply $a = b$ (by A1.1.3), contradicting the assumption that $a \neq b$.

We now turn to the proof of A3.2.7. Let $b'' = P(aba')$, $c'' = P(aca')$, $x'' = P(axa')$ and $y = R(a'F(b'b''a')x'')$. By the proof of A3.2.6 and the uniqueness statement therein, $c' = R(a'F(b'b''a')c'')$ and $b' = R(a'F(b'b''a')b'')$. By (3.39), A3.3.1, (3.36), (3.30), (3.22) $ax \equiv a'x''$, $bx \equiv b''x''$, $cx \equiv c''x''$, $a'x'' \equiv a'y$, $b''x'' \equiv b'y$, $c''x'' \equiv c'y$, and, since we know that $ax \equiv a'x'$ and $bx \equiv b'x'$, we get (using A2.1.2, A2.1.1, A1.1.1) $a'x' \equiv a'y$, $b'x' \equiv b'y$. If $x' = y$, then, since $cx \equiv c''x''$, $c''x'' \equiv c'y$, we get $cx \equiv c'x'$ (by A2.1.2). If $x' \neq y$, then we cannot have both $L(a'b'x')$ and $L(a'b'y)$. For if $L(a'b'x') \wedge L(a'b'y)$, then, by (3.32), we should have $a' = b'$, which in turn would imply $a = b$ (by A1.1.3, since $ab \equiv a'b'$), contradicting $\neg L(abx)$. If $\neg L(a'b'x')$, then $y = R(a'b'x')$ (by A3.3.5), hence $c'x' \equiv c'y$ (by (3.12), A3.3.7). The same conclusion is reached by assuming $\neg L(a'b'y)$. Since we also had $cx \equiv c''x''$, $c''x'' \equiv c'y$, we deduce $cx \equiv c'x'$ (by A2.1.1, A2.1.2). This proves A3.2.7.

The existence statement in A3.2.8 is satisfied with $x' = R(abx)$ (by (3.12), (3.36), (3.20)), and the uniqueness statement holds by A3.3.5.

Since it follows from A3.2.8 that in the antecedent of A3.2.9 $x' = R(abx)$ and $y' = R(aby)$, the consequent follows from A3.3.7.

The existence statement in A3.2.11 is satisfied with $b' = \sigma(ab)$ (by A3.3.6, (3.11),(3.36)), and the uniqueness statement follows from (3.31), A3.2.2).

A3.2.12 holds with $c = P(yab)$. To see this, first notice that $L(xyc)$ (by (3.40), A3.2.2, A3.2.3), $yb \equiv ac$ (by (3.22), A3.3.1), hence $ac \equiv bz$ (by A2.1.2, A2.1.1, A1.1.1) and $ya \equiv bc$ (by (3.22)), hence $ax \equiv bc$ (by A2.1.2, A2.1.1, A1.1.1). Since, according to A3.2.11, $y = \sigma(ax)$, we have $a \neq x$ (by (3.10)), we deduce from A3.2.6 (since $ax \equiv bc$), that there is a z', such that $L(bcz') \wedge ac \equiv bz' \wedge xc \equiv cz'$. If $z' = z$, then $xc \equiv cz$, i. e. $I(czx)$ (by A2.1.2, A2.1.1, A1.1.1), and we are done. If $z' \neq z$, then, since $ac \equiv bz$ and $ac \equiv bz'$, we have $bz \equiv bz'$ (by A2.1.2, A2.1.1), hence $z' = \sigma(bz)$ (by A3.2.11 and the fact that $L(zz'b)$ (by A3.2.2, A3.2.3)). However, $y = \sigma(bz)$ as well (by A3.2.11), hence $z' = y$. This means that $I(cxy)$ so, since we have $I(axy)$ as well, we get $c = a$ (by (3.32)), i. e. $P(yab) = a$, which implies $y = b$ (by A3.3.1, A3.3.2). By A3.2.11 we also have $z = \sigma(by)$, so $y = b$ implies $y = z$ (by (3.10)), a contradiction.

Suppose $L(abc) \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c'$. We shall prove, as in [77, (1.2)], that $L(a'b'c')$. According to A3.2.6, there is a c'', such that $L(a'b'c'') \wedge ac \equiv a'c'' \wedge bc \equiv b'c''$. Under the assumption that $\neg L(a'b'c')$, we infer, using A3.2.7 and A2.1.1, that $c'c'' \equiv cc$, therefore $c' = c''$ (by A1.1.3), hence $L(a'b'c')$, contradicting our assumption. We have thus proved that, for $a \neq b$

$$L(abc) \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c' \rightarrow L(a'b'c'), \tag{3.41}$$

which is true for $a = b$ as well (by A1.1.3, (3.3)). From (3.41) we deduce, using A3.3.7, that

$$a \neq b \to (L(xyz) \to L(R(abx)R(aby)R(abz))). \tag{3.42}$$

Since $ba \equiv cP(bac)$ and $bc \equiv aP(bac)$ (by (3.22), A3.3.1), we get, using (3.41), A3.2.1-A3.2.4, A2.1.2, A2.1.1, A1.1.1 (and, for $a = c$, A3.3.6, (3.1)),

$$L(abc) \leftrightarrow L(aP(bac)b) \tag{3.43}$$

A proof of

$$x \neq x' \wedge I(axx') \wedge I(bxx') \wedge I(cxx') \to L(abc) \tag{3.44}$$

based on the axioms A3.2.1-A3.2.11, A1.1.1, A1.1.3, and $(\exists abc)\neg L(abc)$ (which is a consequence of A3.2.14, A3.2.4-A3.2.5, A1.1.1, A1.1.3, A3.2.6) can be found in [77, (1.5)].

We now turn to the proof of

$$L(abF(abx)). \tag{3.45}$$

Let $\neg L(abx)$, $o = F(abx)$ and suppose that $\neg L(abo)$. By A3.2.2, A3.2.3, we cannot have both $L(xoa)$ and $L(xob)$ (since $o \neq x$ (by (3.3))), so we may assume w. l. o. g. that $\neg L(xoa)$. Let $x' = R(abx)$ and $o' = R(abo)$. Since $x' = \sigma(ox)$, we get $I(oxx')$ (by (3.36)). We also have $ox \equiv o'x'$ (by A3.3.7), hence $I(x'oo')$ (by A2.1.2, A2.1.1, A1.1.1). By (3.36) we also have $I(axx')$ and $I(aoo')$. Let $r = R(xx'a)$. Since $L(xx'o)$ (by A3.3.6, A3.2.2) and $L(xx'o')$ (by A3.2.2, since $L(xx'o) \to L(x'xo')$ (by (3.42), (3.33)), so $R(xx'a) = R(oo'a)$ (by A3.3.4, (3.3), (3.2) and the fact that $x \neq x'$ and $o \neq o'$ (by (3.12))). Since r is both $R(xx'a)$ and $R(oo'a)$, we deduce from (3.36) that $I(xar)$, $I(x'ar)$, $I(oar)$, $I(o'ar)$. Using A2.1.2, A2.1.1, A1.1.1 we get $I(rxx')$ and $I(roo')$. From $x \neq x' \wedge I(axx') \wedge I(rxx') \wedge I(oxx')$ we conclude, using (3.44), that $L(aro)$. From $x \neq x' \wedge I(aoo') \wedge I(roo') \wedge I(x'oo')$ we conclude, using (3.44), that $L(arx')$, hence we have $L(aox')$ (by A3.2.3, since $a \neq r$, which we derive by (3.12), bearing in mind that $r = R(xx'a) = R(xoa)$ (by A3.3.4, (3.3), (3.2))). Since we also have $L(xx'o)$, we deduce $L(xoa)$ (by A3.2.2, A3.2.3 and the fact that $o \neq x'$ (by (3.11))). We have arrived at a contradiction, wherefrom we conclude (3.45) in case $\neg L(abx)$. If $L(abx)$, then, according to (3.3), $a = b$ or $F(abx) = x$, so (3.45) holds.

A3.2.10 holds with $y = F(abx)$ (by A3.3.6, (3.45), (3.2), A3.2.2, since, by A3.2.8, $x' = R(abx)$).

Let $a \neq c$, $L(abc)$ and $b' = P(P(bac)ca)$. We want to prove that $b' = b$. Then, by (3.22) and A3.3.1, $bc \equiv aP(bac)$, $ba \equiv cP(bac)$, $P(bac)a \equiv cb'$, $P(bac)c \equiv ab'$, hence by A2.1.2, A2.1.1, A1.1.1, $I(cbb')$, $I(abb')$. We also have $L(bb'a)$ and $L(bb'c)$ (by (3.43) and A3.2.2, A3.2.3), hence, if $b' \neq b$, then, applying (3.32), we get $a = c$, a contradiction. Hence, for $a \neq c \wedge L(abc)$, we have proved $P(P(bac)ca) = b$ (which is true for $a = c$ as well (by (3.9), (3.1))), i. e.

$$L(abc) \to P(P(bac)ca) = b. \tag{3.46}$$

Let $\neg L(abc)$ and $c' = P(\sigma(ab)aP(bac))$. We want to prove that $P(P(bac)ca) = b$. Since $P(cP(bac)a) = \sigma(ab)$ (by (3.29)), using (3.22) and A3.3.1, we have $ca \equiv P(bac)\sigma(ab)$, $cP(bac) \equiv a\sigma(ab)$, $\sigma(ab)a \equiv P(bac)c'$, $\sigma(ab)P(bac) \equiv ac'$. Using A2.1.2, A2.1.1, A1.1.1 we get $I(acc')$, $I(P(bac)cc')$. Since $P(P(bac)c'a) = \sigma(a\sigma(ab))$ (by (3.29)) and $\sigma(a\sigma(ab)) = b$ (by (3.9)), we have $P(P(bac)c'a) = b$, so $P(bac)a \equiv c'b$ (by (3.22), A3.3.1). Together with

$bc \equiv aP(bac)$ (by (3.22), A3.3.1), this implies, by A2.1.2, A2.1.1, A1.1.1, $I(bcc')$. Since $I(acc')$, $I(P(bac)cc')$, $I(bcc')$, if $c' \neq c$, then we deduce from (3.44) that $L(aP(bac)b)$, hence $L(abc)$ (by (3.43)), a contradiction. Therefore $c' = c$, which means

$$\neg L(abc) \to P(\sigma(ab)aP(bac)) = c. \tag{3.47}$$

By (3.29) we have $P(P(bac)P(\sigma(ab)aP(bac))a) = \sigma(a\sigma(ab))$. The LHS is $P(P(bac)ca)$ (by (3.47)), and the RHS is b (by (3.9)), hence

$$\neg L(abc) \to P(P(bac)ca) = b, \text{ therefore} \tag{3.48}$$

$$P(P(bac)ca) = b \text{ (by (3.48), (3.46)).} \tag{3.49}$$

$$P(bac) = \sigma(cP(abc)) \text{ (by (3.49), (3.29)).} \tag{3.50}$$

We now want to prove that

$$\sigma(P(abc)\sigma(ca)) = \sigma(ba). \tag{3.51}$$

Since, by (3.29) and A3.3.1), $P(bP(abc)c) = \sigma(ca)$, (3.49) implies $P(\sigma(ca)cP(abc)) = b$. From $P(cxP(ycx)) = \sigma(xy)$ (by (3.29), A3.3.1) with $x = P(abc)$ and $y = \sigma(ca)$, we deduce $P(cP(abc)b) = \sigma(P(abc)\sigma(ca))$ (since $P(ycx) = b$), which, using (3.29), gives the desired equality.

We now turn to the proof of A3.2.13. By A3.2.11, A3.2.13 is equivalent to

$$\neg L(xyz) \wedge \sigma(ax) = y \wedge \sigma(bz) = y \wedge \sigma(cx) = z \to \neg L(abc). \tag{3.52}$$

Suppose now x, y, z, a, b, c are as in the antecedent of (3.52). First note that we have $\neg L(xac)$ (as $L(xac)$ would, together with $L(cx\sigma(cx))$ and $L(ax\sigma(ax))$ (by A3.3.6), imply $L(x\sigma(ax)\sigma(cx))$ (by A3.2.2, A3.2.3, (3.10)), i. e. $L(xyz)$). We deduce from (3.51) that $\sigma(P(xac)\sigma(cx)) = \sigma(ax)$, i. e. $\sigma(P(xac)z) = y$, which, together with $\sigma(bz) = y$ implies $P(xac) = b$ (by A3.3.3). Therefore $L(abc)$ would mean $L(aP(xac)c)$, which would imply that $L(xac)$ (by (3.43), A3.2.2, A3.2.3), so we must have $\neg L(abc)$.

Suppose $ab \equiv a'b'$ and $a \neq b$. By A3.2.6, there is a unique x such that $L(a'b'x) \wedge b\sigma(ba) \equiv b'x \wedge a\sigma(ba) \equiv a'x$. From $I(ba\sigma(ba)) \wedge I(b'a'\sigma(b'a'))$ (by (3.36)) we deduce, using A2.1.2, A2.1.1, A1.1.1, that $I(b'xa')$. Since we also have $L(a'b'x)$, we conclude, using A3.2.11, that x is either a' or $\sigma(b'a')$. Since $x = a'$ would imply $a = \sigma(ba)$ (by A1.1.3), i. e. $a = b$ (by (3.11)), we must have $x = \sigma(b'a')$, hence

$$ab \equiv a'b' \to a\sigma(ba) \equiv a'\sigma(b'a'), \tag{3.53}$$

which is seen to be true for $a = b$ as well (by A1.1.3, (3.10), A3.2.5). We now define the notion of perpendicularity, which we shall use, for improved readability, in the proof of A3.2.14.

$$ab \perp ac \overset{\text{def}}{\leftrightarrow} F(abc) = a. \tag{3.54}$$

Note that $ab\perp ac \to ac\perp ab$ (by (3.26), (3.18)),

$$a \neq b \wedge ab\perp ac \wedge L(abx) \wedge x \neq a \to ax\perp ac \text{ (by A3.3.4, (3.3)),} \tag{3.55}$$

and that A3.2.14 states that there are four points, denoted by a_1, a_2, a_3, a_4, such that $a_i a_{i-1} \perp a_i a_{i+1}$ (summation in the indices being mod 4) (by (3.26), (3.36), (3.2), A2.1.2, A2.1.1, A1.1.1). We shall construct a rectangle starting from the three given points a_0, a_1, a_2, although there is nothing special about them; we could have started with any three noncollinear points a, b, c, and then work with $F(abc)$, c and either a or b (depending on which is different from $F(abc)$) instead of a_0, a_1, a_2. We shall show that, if $F(a_0a_1a_2) \neq a_1$, then a_0', a_1, a_4, a_2 form the vertices of a rectangle, where $a_0' = F(a_0a_1a_2)$ and $a_4 = P(a_0'a_1a_2)$ (if $F(a_0a_1a_2) = a_1$, then an analogous proof shows that a_0', a_0, a_4', a_2 form the vertices of a rectangle, with $a_4' = P(a_0'a_0a_2)$). To simplify notation, let $a_3 = \sigma(a_1a_0')$, $a_5 = \sigma(a_0'a_2)$, $a_6 = \sigma(a_1a_4)$. We have $P(a_2a_4a_1) = a_3$ (by (3.29)), $a_0'a_2 \equiv a_1a_4$ (by (3.22), A3.3.1), hence $a_2a_5 \equiv a_4a_6$ (by (3.53), A2.1.2, A2.1.1, A1.1.1). We can now apply A3.2.7 (using (3.36) and A2.1.2, A2.1.1, A1.1.1 as well) to deduce $a_5a_1 \equiv a_6a_3$. This implies, since $a_2a_1 \equiv a_4a_3$ (by (3.29), (3.22), A3.3.1) and $I(a_1a_2a_5)$ (since $R(a_0'a_1a_2) = a_5$ (by (3.2)), using (3.36)), that $I(a_3a_4a_6)$ (by A2.1.2, A2.1.1, A1.1.1), hence $a_6 = R(a_1a_3a_4)$ (by A3.3.5, since $\neg L(a_1a_3a_4)$ (by (3.43), A3.3.6, A3.2.2, A3.2.3)), i. e. $a_6 = \sigma(F(a_1a_3a_4)a_4)$, and, since $a_6 = \sigma(a_1a_4)$ as well, we get $F(a_1a_3a_4) = a_1$, i. e. $a_1a_3 \perp a_1a_4$, hence $a_1a_0' \perp a_1a_4$ (by (3.55), A3.3.6).

We have thus proved that, if we know that the parallelogram $a_0'a_1a_4a_2$ has a right angle at a_0', then it has one at a_1 as well. We can now let a_1 play the role of a_0' and reach the conclusion that $a_4a_1 \perp a_4a_2$ (since we have $P(a_1a_4a_0') = a_2$ (by (3.27), A3.3.1)), and similarly with a_4 playing the role of a_0' to get $a_2a_4 \perp a_2a_0'$ (since $P(a_4a_2a_1) = a_0'$ (by (3.49))). This proves A3.2.14. One easily sees that, replacing A3.3.11 by A3.3.12, one can deduce A3.2.15.

By the representation theorem for $\mathcal{ME}$, we know what the universe of a model of $\mathcal{CME}$ is, and that L and $\equiv$, as defined by (3.3) and (3.5). are to be interpreted as the collinearity and the Euclidean congruence relation respectively. This in turn implies, by A3.2.11, that $\sigma(\mathbf{ab})$ is the point obtained by reflecting $\mathbf{b}$ in $\mathbf{a}$, and by A3.2.8, that $\mathbf{R}(\mathbf{abc})$ is the reflection of $\mathbf{c}$ in the line $\mathbf{ab}$ whenever $\mathbf{a} \neq \mathbf{b}$. This, together with (3.2), implies that $\mathbf{F}(\mathbf{abc})$ is the footpoint of the perpendicular from $\mathbf{c}$ to the line $\mathbf{ab}$ whenever $\mathbf{a} \neq \mathbf{b}$. For $\mathbf{a} = \mathbf{b}$, (3.13) tells us that $\mathbf{F}(\mathbf{aab}) = \mathbf{a}$, so we have obtained an algebraic description of $\mathbf{F}(\mathbf{abc})$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

From (3.22) and (3.29) we deduce that $\mathbf{x} = \mathbf{P}(\mathbf{abc})$ satisfies
$\|\mathbf{b}-\mathbf{x}\| = \|\mathbf{a}-\mathbf{c}\|, \|\mathbf{c}-\mathbf{x}\| = \|\mathbf{a}-\mathbf{b}\|, \|(2\mathbf{b}-\mathbf{a})-\mathbf{x}\| = \|\mathbf{b}-\mathbf{c}\|$.
Solving this system of equations, we get (for both collinear and non-collinear $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$) $\mathbf{P}(\mathbf{abc}) = \mathbf{b}-\mathbf{a}+\mathbf{c}$. We thus have the following

Representation Theorem 3.3.1 *$\mathfrak{M} \in Mod(\mathcal{CME})$ iff $\mathfrak{M} \simeq \langle E, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{F}_{(K,k)}, \mathbf{P}_K\rangle$, with E as in the Representation Theorem 3.2.1, $\mathbf{a}_0 = (0,0)$, $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (\alpha,\beta)$, with α, β in K, $\beta \neq 0$, $\mathbf{F}_{(K,k)}(\mathbf{abc}) = \mathbf{f} = f_1 + f_2\sqrt{-k}$, for $\mathbf{a} \neq \mathbf{b}$, where*
$f_1 = ((c_1(a_1-b_1)^2 + (a_1-b_1)(a_2-b_2)c_2 + k(a_2-b_2)(a_2b_1-a_1b_2))(\|\mathbf{a}-\mathbf{b}\|)^{-1}$,
$f_2 = ((kc_2(a_2-b_2)^2 + (a_1-b_1)(a_2-b_2)c_1 - (a_1-b_1)(a_2b_1-a_1b_2))(\|\mathbf{a}-\mathbf{b}\|)^{-1}$,
and $\mathbf{F}_{(K,k)}(\mathbf{aac}) = \mathbf{a}$; $\mathbf{P}_K(\mathbf{abc}) = \mathbf{b}-\mathbf{a}+\mathbf{c}$.
$\mathfrak{M} \in Mod(\mathcal{CME}')$ iff $\mathfrak{M} \simeq \langle E, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{F}_{(K,1)}, \mathbf{P}_K\rangle$, with E as in the Representation Theorem 3.2.1 (with $k=1$), $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{F}_{(K,1)}, \mathbf{P}_K$ as above, with $\alpha = 0$, $\beta = 1$.

3.4 Rectangular planes

Rectangular planes were introduced in [38] and axiomatized by R. Stanik in [78] in a two-sorted first-order language, with variables for 'points' and 'lines', with three primitive notions 'incidence' ($\dot{\in}$) — a binary predicate with a point in the first argument and a line in the second — 'parallelity' ($\|$) — a binary predicate among lines — and 'congruence' ($\equiv$) — a quaternary predicate among points. As shown in [72, II. 4.59], we can rephrase her axiom system in terms of points only. With lines written as $\langle x, y\rangle$ with $x \neq y$, we introduce the following predicates among points:
$L(xyz) \overset{def}{\leftrightarrow} z \dot{\in} \langle x, y\rangle \vee x = y$,
$xy\|x'y' \overset{def}{\leftrightarrow} x \neq y \wedge x' \neq y' \wedge \langle x, y\rangle \| \langle x', y'\rangle$,
which, together with $\equiv$, allow us to restate Stanik's axioms with point-variables only. The axioms are: A3.2.1-A3.2.5, A1.1.1, A1.1.3, A3.2.8,

A 3.4.1 $ab\|cd \rightarrow a \neq b \wedge c \neq d,$

A 3.4.2 $c \neq e \wedge ab\|cd \wedge L(cde) \rightarrow ab\|ce,$

A 3.4.3 $a \neq b \rightarrow ab\|ab \wedge ab\|ba,$

A 3.4.4 $ab\|cd \rightarrow cd\|ab,$

A 3.4.5 $ab\|cd \wedge cd\|ef \rightarrow ab\|ef,$

A 3.4.6 (i) $(\forall pxy)(\exists u)\, [x \neq y \rightarrow pu\|xy],$
(ii) $ab\|cd \wedge ab\|ce \rightarrow L(cde),$

A 3.4.7 $(\forall abc)(\exists d)\, [\neg L(abc) \rightarrow ad\|bc \wedge cd\|ab],$

A 3.4.8 $(\forall oabc)(\exists uvw)\, [\neg L(oab) \wedge \neg L(oac) \wedge \neg L(ocb) \rightarrow u \neq o \wedge v \neq o$
$\wedge\, w \neq o \wedge L(oau) \wedge L(obv) \wedge L(ocw) \wedge L(uvw)],$

A 3.4.9 $\neg L(abc) \wedge ab\|cd \rightarrow (ab \equiv cd \leftrightarrow ac\|bd \vee ad\|bc),$

A 3.4.10 $(\forall abcd)(\exists e)\, [\neg L(abc) \wedge ab \equiv ac \wedge L(abd) \rightarrow L(ace) \wedge de\|bc \wedge ad \equiv ae],$

A 3.4.11 $L(abc) \wedge \neg L(abd) \wedge ad \equiv ad' \wedge bd \equiv bd' \rightarrow cd \equiv cd',$

A 3.4.12 $(\exists abcd)\, [\neg L(abc) \wedge ad\|bc \wedge cd\|ba \wedge \neg ac\|bd].$

Notice that we could have chosen to express the axioms in a language that contains only $\|$ and $\equiv$ as primitive notions, since L can be defined in terms of $\|$ by

$$L(abc) \leftrightarrow a = b \vee a = c \vee ab\|ac, \tag{3.56}$$

i. e. (3.56) can be deduced from A3.4.2, A3.4.3, A3.4.6(ii), A3.2.1-A3.2.3.

Let $\Gamma = \{$A3.2.1-A3.2.5, A1.1.1, A1.1.3, A3.2.8, A3.4.1-A3.4.12$\}$, $\Gamma' = \Gamma \cup \{$A3.2.15$\}$, $\mathcal{RE} \overset{\text{def}}{=} Cn(\Gamma)$, $\mathcal{RE}' \overset{\text{def}}{=} Cn(\Gamma')$. R. Stanik [78], [38], [36, III.6] proved the following algebraic characterization of models of $\mathcal{RE}$ (from which the one for $\mathcal{RE}'$ easily follows)

Representation Theorem 3.4.1 *$\mathfrak{M} \in Mod(\mathcal{RE})$ iff $\mathfrak{M} \simeq \langle E, \mathbf{L}_K, \equiv_{(K,k)}, \|_K\rangle$, where K is a field of characteristic $\neq 2$, $k \in K$, $-k \notin K^2$, $L = K(\sqrt{-k})$, $\|z\| = x^2+ky^2$ for $z = x+y\sqrt{-k}$, $L_1 = \{z \in L \| \|z\| = 1\}$, $R = R(K,k)$ the subring of L generated by L_1, $E \subset K$ is an R-module with $0, 1 \in E$, $\mathbf{L}_K$ and $\equiv_{K,k}$ are as in Representation Theorem 3.2.1, and $\mathbf{xy}\|_K\mathbf{uv}$ iff $\mathbf{x} \neq \mathbf{y}, \mathbf{u} \neq \mathbf{v}, K(\mathbf{x}-\mathbf{y}) = K(\mathbf{u}-\mathbf{v})$, with $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in E$. $\mathfrak{M} \in Mod(\mathcal{RE}')$ iff $\mathfrak{M} \simeq \langle E, \mathbf{L}_K, \equiv_{(K,1)}, \|_K\rangle$, with notations as above, with $k = 1$.*

In other words, the point-set of a rectangular plane of characteristic $\neq 2$ (i. e. universe of a model of Γ) is isomorphic to a subset E of $L = K(\sqrt{-k})$, which satisfies
(i) $(E, +)$ is a subgroup of $(L, +)$ and $1 \in E$,
(ii) $(\forall s \in L)\|s\| = 1 \Rightarrow s \cdot E \subseteq E$.
It inherits the collinearity, congruence and parallelity relations from the Euclidean plane over (K, k) (or, equivalently over (L, K)) (cf. [54]).

This means, geometrically speaking, that this point-set contains 0, 1 and is closed under translations and rotations around 0.

Rectangular planes of characteristic $\neq 2$ were defined in [38] as non-Fanoian (i. e. satisfying A3.4.12) Abelian translation structures $\mathfrak{A}$, such that for every line g in $\mathfrak{A}$ there is an involutory automorphism of $\mathfrak{A}$ whose fixpointset is g (called the reflection in l) and such that the three-reflection theorem holds for concurrent lines.

Abelian translation structures have been defined by J. André in [1] and [2] by axioms (a 0)-(a 3), (t 1)-(t 2) and 'All translations are central'. We shall restate these axioms in a language with the primitive notions L, $\|$ and the ternary operation P, where $P(abx)$ may be read as 'the image of x under the translation that moves a into b'; we shall write $\tau_{ab}(x)$ for $P(abx)$. The axioms for Abelian translation structures are A3.2.1-A3.2.3, A3.4.1-A3.4.6, A3.4.13-A3.4.18, where

A 3.4.13 $(\exists abc)\neg L(abc)$,

A 3.4.14 $\tau_{ab}(a) = b$,

A 3.4.15 $\tau_{ab}(x) = x \leftrightarrow a = b$,

A 3.4.16 $p \neq q \to pq\|\tau_{ab}(p)\tau_{ab}(q)$,

A 3.4.17 $a \neq b \to p\tau_{ab}(p)\|q\tau_{ab}(q)$,

A 3.4.18 $\tau_{cd}(\tau_{ab}(x)) = \tau_{a\tau_{cd}(b)}(x)$.

A3.4.14 states that translations (i. e. the τ's) act transitively; A3.4.15 states that proper translations (i. e. $\neq$ identity) don't have fixed points; A3.4.16 states that translations are morphisms of the parallelity structure (hence automorhisms); A3.4.17 states that translations are 'central', and A3.4.18 states that the translations form a group under composition (where τ_{aa} is the identity element and where τ_{ba} is the inverse of τ_{ab} (by A3.4.14, A3.4.15, A3.4.18)). It follows from A3.4.14, A3.4.15 and A3.4.18 that $\tau_{ba}(\tau_{ab}(x)) = \tau_{a\tau_{ba}(b)}(x)$; this implies that translations are bijections.

In order to state the axioms on the existence of an involutory automorphism of an Abelian structure satisfying the conditions stated earlier, we need to enlarge the language with another ternary operation R', where $R'(abx)$ may be read as 'the reflection of x in the line ab if $a \neq b$, arbitrary, otherwise'. We shall write $\rho_{ab}(x)$ for $R'(abx)$. The axioms we need to add to those of a non-Fanoian Abelian translation structure in order to obtain an axiomatization of non-Fanoian rectangular planes are

A 3.4.19 $a \neq b \wedge c \neq d \wedge L(abc) \wedge L(abd) \rightarrow \rho_{ab}(x) = \rho_{cd}(x),$

A 3.4.20 $a \neq b \rightarrow (\rho_{ab}(x) = x \leftrightarrow L(abx)),$

A 3.4.21 $a \neq b \wedge xy \| uv \rightarrow \rho_{ab}(x)\rho_{ab}(y) \| \rho_{ab}(u)\rho_{ab}(v),$

A 3.4.22 $a \neq b \rightarrow \rho_{ab}(\rho_{ab}(x)) = x,$

A 3.4.23 $(\forall a_1a_2a_3b_1b_2b_3o)(\exists ab)(\forall x) \bigwedge_{i=1}^{3}(a_i \neq b_i \wedge L(oa_ib_i)) \rightarrow a \neq b \wedge L(oab)$
$\wedge\, \rho_{a_1b_1}(\rho_{a_2b_2}(\rho_{a_3b_3}(x)))) = \rho_{ab}(x).$

A3.4.19 states, that there is at most one reflection in a given line; A3.4.20 that the fixpointset of a reflection in the line determined by a and b is that very line, and hence that the correspondence between reflections and lines in one-to-one; A3.4.22 that reflections are involutory (and one can deduce from it that reflections are bijections); A3.4.21 that reflections are morhisms (hence automorphisms), and A3.4.23 is the three-reflection axiom, stating that the composition of three reflections in three concurrent lines is a reflection in a line that passes through the common point of the first three.

Let $\Gamma_0 = \{$A3.2.1-A3.2.3, A3.4.1-A3.4.6, A3.4.12-A3.4.23$\}$. The following representation theorem was proved in [38]

Representation Theorem 3.4.2 *$\mathfrak{M} \in Mod(\Gamma_0)$ iff $\mathfrak{M} \simeq \langle E, \mathbf{L}_K, \|_K, \mathbf{P}_K, \mathbf{R}'_{(K,k)}\rangle$, where E, $\mathbf{L}_{(K,k)}$, $\|_{(K,k)}$ are as in the Representation Theorem 3.4.1 and* $\mathbf{P}_K(\mathbf{abx}) = \mathbf{b} - \mathbf{a} + \mathbf{x}$;
$\mathbf{R}'_{(K,k)}(\mathbf{abx}) = \mathbf{r} = r_1 + r_2\sqrt{-k}$, *for* $\mathbf{a} \neq \mathbf{b}$ *where*
$r_1 = (k(2a_1 - x_1)(b_2 - a_2)^2 + x_1(b_1 - a_1)^2 + 2k(b_2 - a_2)(x_2 - a_2)(b_1 - a_1))(\|\mathbf{b} - \mathbf{a}\|)^{-1}$,
$r_2 = ((2a_2 - x_2)(b_1 - a_1)^2 + kx_2(b_2 - a_2)^2 + 2(b_1 - a_1)(x_1 - a_1)(b_2 - a_2))(\|\mathbf{b} - \mathbf{a}\|)^{-1}$,
arbitrary, for $\mathbf{a} = \mathbf{b}$, *with* $\mathbf{a}, \mathbf{b}, \mathbf{x} \in E$.

3.5 A constructive axiom system for rectangular planes

In this section, we shall provide a quantifier-free axiom system for rectangular planes in a language L_{re} =L(a_0, a_1, a_2, P, R), where a_0, a_1, a_2 are individual constants (standing for three non-collinear points), P and R ternary operations, to be read as $P(abc) = d$ iff '$abdc$ is a parallelogram, i.e. $ab\|cd \wedge ac\|bd$' and $R(abc) = d$ iff 'd is the reflection of c in the line ab, if $a \neq b$; d is the reflection of c in a if $a = b$'.

In stating the axioms we shall use the abbreviations

$$L(abc) \overset{\text{def}}{\leftrightarrow} R(abc) = c \vee a = b, \tag{3.57}$$

$$I(abc) \overset{\text{def}}{\leftrightarrow} R(aR(cba)b) = c, \tag{3.58}$$

(3.1) and (3.5). With

A 3.5.1 $a \neq b \wedge c \neq d \wedge R(abc) = c \wedge R(abd) = d \rightarrow R(abx) = R(cdx)$,

A 3.5.2 $a_0 \neq a_1 \wedge a_0 \neq a_2 \wedge a_1 \neq a_2 \wedge R(a_0a_1a_2) = \sigma(a_0a_2) \wedge I(a_0a_1a_2)$,

let $\mathcal{CRE} = Cn_{\mathrm{L}_{re}}$(A3.2.1, A3.3.1-A3.3.3, A3.3.5-A3.3.11, A3.5.1, (3.19)) and $\mathcal{CRE}' = Cn_{\mathrm{L}_{re}}$(A3.2.1, A3.3.1-A3.3.3, A3.3.5-A3.3.10, A3.5.1, A3.5.2, (3.19)), which are considered to be L_{re} theories. Let parallelity, the translation and reflection operations be defined by

$$ab \| cd \overset{\text{def}}{\leftrightarrow} a \neq b \wedge c \neq d \wedge L(cdP(bac)), \tag{3.59}$$

$$a \neq b \wedge R'(abx) = y \overset{\text{def}}{\leftrightarrow} a \neq b \wedge R(abx) = y. \tag{3.60}$$

Let $\Phi = \{$A3.2.1, A3.3.1-A3.3.3, A3.5.1, A3.3.5-A3.3.11 (3.19), (3.1), (3.57), (3.58), (3.5), (3.59), (3.60)$\}$. In order to show that the models of $\mathcal{CRE}$ are rectangular planes, we shall prove that $\Phi \vdash \Gamma_0$. Our task is now greatly simplified by the fact that the axiom system for $\mathcal{CRE}$ has many axioms which it either shares with $\mathcal{CME}$ or which are theorems proved in $\mathcal{CME}$. Many proofs of theorems in $\mathcal{CME}$ can be repeated verbatim; for some the only difference is that F has to be replaced everywhere by R.

$ab \equiv ac \leftrightarrow I(abc)$ is true in this case as well. Without a change in the proofs we get (3.6), (3.7), (3.8), (3.9), (3.10), (3.11). Just by replacing F with R in the proofs carried out in §3, we get (3.17), (3.20). Since, by A3.5.1, $a \neq b \wedge R(aba) = a \wedge R(ab\sigma(ab)) = \sigma(ab) \rightarrow R(abb) = R(a\sigma(ab)b)$ is true, and since the antecedent is true (by A3.2.1, (3.57), A3.3.6) and $R(abb) = b$ (by (3.17), (3.57)), we get $R(a\sigma(ab)b) = b$ for $a \neq b$ which is seen to be true for $a = b$ as well (by (3.19), (3.10)), hence we get $I(abb)$ (by (3.58)), i. e. A3.2.4 is true. (3.22), (3.23), A3.2.2, A3.2.3, A2.1.2, (3.27), (3.33) can be derived in exactly the same way, whereas (3.28) follows from (3.33) and (3.20). A3.2.5, i. e. $I(bP(aab)b)$ follows from (3.7), (3.58), (3.19), (3.10). Since $R(a\sigma(ab)b) = b$, we get, by using (3.19), (3.9), $R(bR(a\sigma(ba)b)\sigma(ba)) = R(bb\sigma(ba)) = \sigma(b\sigma(ba)) = a$, hence $I(bP(abb)a)$ (by (3.58)), i. e. A1.1.1 is true.

Let $ab \equiv cc$, i. e. $I(cP(bac)c)$, that is $R(cR(cP(bac)c)P(bac)) = c$ (by (3.58). We know that $R(cP(bac)c) = c$ (by A3.2.1, (3.57), (3.19), (3.10)), so our hypothesis becomes $R(ccP(bac)) = c$, i. e. $\sigma(cP(bac)) = c$ (by (3.19)), and this implies $P(bac) = c$ (by (3.11)), wherefrom we conclude that $a = b$ (by A3.3.2), thus proving A1.1.3. The proofs of (3.29) and (3.30) remain unchanged. To prove (3.31), suppose its antecedent is true, i. e. $a \neq b \wedge L(abx) \wedge I(xab)$. By (3.58) we get $R(xR(bax)a) = b$ and, by (3.57) and A3.2.2, $R(bax) = x$, hence $R(xxa) = b$, i. e. $\sigma(xa) = b$ (by (3.19)), proving that (3.31) remains true.

The proofs of (3.32) and (3.35), (3.36), (3.38), (3.39), (3.40) remain unchanged. The only change occurring in the proofs of A3.2.6 and A3.2.7 is the replacement of F by R throughout those proofs; the proofs of A3.2.8, A3.2.9, A3.2.11, (3.41), (3.43), (3.44), (3.49), (3.42) need only minor changes in the reason why a certain formula is true (changes caused by the changed definitions of L and I), but the proof lines remain unchanged.

A3.4.13 is true with $a = a_0$, $b = a_1$ $c = a_2$ (by A3.3.11).

A3.4.1 is true by (3.59).

$ab \| ab$ holds iff $a \neq b$ and $L(abP(baa))$ (by (3.59)), i. e. iff $L(ab\sigma(ab))$ (by (3.1)), which is true (being A3.3.6). $ab \| ba$ holds iff $a \neq b$ and $L(baP(bab))$ (by (3.59)), i. e. iff $L(baa)$ (by (3.7)), which is true (by (3.17)). We have thus proved A3.4.3.

Let $ab\|cd$ and $cd\|ef$, i. e. $a \neq b$, $c \neq d$, $e \neq f$, $L(cdP(bac))$ and $L(efP(dce))$. From (3.40) applied to $L(cdP(bac))$ we get $L(eP(cde)P(cP(bac)e))$, i. e. $L(e\sigma(eP(dce))P(bae))$ (since $P(cde) = \sigma(eP(dce))$ (by (3.50)) and $P(cP(bac)e) = P(bae)$ (by A3.3.8)), i. e. $L(eP(dce)P(bae))$ (by A3.3.6, A3.2.2, A3.2.3). Since we also know that $L(efP(dce))$, we conclude that $L(efP(bae))$ (by A3.2.2, A3.2.3 and the fact that $P(dce) \neq e$ (by A3.3.2)), which means that $ab\|ef$ (by (3.59)), hence proves A3.4.5.

Let $ab\|cd$, i. e. $a \neq b \wedge c \neq d \wedge L(cdP(bac))$ (by (3.59)). Using (3.40) we get $L(aP(cda)P(cP(bac)a))$, i. e. , since $P(cP(bac)a) = \sigma(ab)$ (by (3.29)), $L(aP(cda)\sigma(ab))$. Using A3.2.2, A3.2.3, (3.11), A3.3.6, we get $L(abP(cda))$, i. e. $dc\|ab$. Using A3.4.5, we deduce from $cd\|dc$ (by A3.4.3) and $dc\|ab$ that $cd\|ab$, which proves A3.4.4.

A3.4.6(i) holds with $u = P(ypx)$, since, for $x \neq y$, $pu\|xy$ is equivalent with $u \neq p$ (which is true by A3.3.1, A3.3.2) and $L(xyP(upx))$ (which is true since $P(upx) = y$ (by (3.49), A3.3.1) and $L(xyy)$ is true (by (3.17))).

If $ab\|cd$ and $ab\|ce$, i. e. $a \neq b$, $c \neq d$, $c \neq e$, $L(cdP(bac))$ and $L(ceP(bac))$, then $L(cde)$ (by A3.2.2, A3.2.3 and the fact that $P(bac) \neq c$ (by A3.3.2)), hence we proved A3.4.6(ii).

A3.4.14 is (3.7); A3.4.15 is true by A3.3.2 and (3.7).

Let $p \neq q$. Then, since $P(pqP(abp)) = P(abq)$ (by A3.3.8, A3.3.1), we have $L(P(abp)P(abq)P(pqP(abp)))$ (by (3.17)), hence also $qp\|P(abp)P(abq)$ (by (3.59)), from which we derive, using A3.4.3 and A3.4.5, $pq\|P(abp)P(abq)$, which proves A3.4.16.

Let $a \neq b$. Then $p \neq P(abp)$ and $q \neq P(abq)$ (by A3.3.2) and $L(qP(abq)P(pP(abp)q))$ (by (3.17), since $P(pP(abp)q) = P(abq)$ (by A3.3.8)), i. e. $P(abp)p\|qP(abq)$ (by (3.59)), which proves A3.4.17 (by A3.4.3, A3.4.5).

According to A3.3.8 we have both $P(bP(cdb)P(abx)) = P(cdP(abx))$ and $P(bP(axb)P(cdb)) = P(axP(cdb))$, and since the left hand sides of these equations are equal (which follows from applying A3.3.1 twice), we get $P(axP(cdb)) = P(cdP(abx))$, i. e. $P(aP(cdb)x) = P(cdP(abx))$ (by A3.3.1), which is A3.4.18.

In order to prove that the R'-axioms hold, we must show that they hold with $R'(abx)$ replaced by $R(abx)$ (by (3.60) and the fact that the R's refer only to the case when $a \neq b$ in $R'(abx)$).

A3.4.19 follows from A3.5.1 and (3.57) whereas A3.4.20 follows from (3.57).

In order to prove A3.4.21, we shall first show that

$$a \neq b \rightarrow P(R(abx)R(aby)R(abz)) = R(abP(xyz)), \tag{3.61}$$

i. e. that reflections in lines preserve the operation P.

Let $a \neq b$, $\neg L(xyz)$, $P(xyz) = p$, $R(abx) = x'$, $R(aby) = y'$, $R(abz) = z'$, $R(abp) = p'$, $\sigma(yx) = s$, $R(abs) = s'$, $P(x'y'z') = p''$. From (3.42) we deduce that $L(x'y's')$ and from A3.3.7 that $xy \equiv x'y'$, $ys \equiv y's'$. Since we also have $I(yxs)$ (by (3.36)), we conclude that $I(y'x's')$ (by A2.1.2, A2.1.1, A1.1.1), hence that $s' = \sigma(y'x')$ (by A3.2.2, (3.31)). Suppose $p' \neq p''$. Since $ps \equiv p's'$, $pz \equiv p'z'$, $zy \equiv z'y'$ (by A3.3.7), and since $z'y' \equiv p''s'$, $zy \equiv ps$ (by (3.22), (3.29), A3.3.1), we have $ps \equiv p''s'$ (by A2.1.2, A2.1.1). Bearing in mind that $ps \equiv p's'$ (by A3.3.7), we conclude that $I(s'p'p'')$ (by A2.1.2, A2.1.1, A1.1.1). From (3.22), A3.3.1 we get $xz \equiv yp$, $x'z' \equiv y'p''$ and from A3.3.7 we get $xz \equiv x'z'$, $yp \equiv y'p'$, allowing us to conclude that $I(y'p'p'')$ (by A2.1.2, A2.1.1). The congruences $zp \equiv z'p'$, $xy \equiv x'y'$ (by A3.3.7) and $xy \equiv zp$, $x'y' \equiv z'p''$ (by (3.22)) imply that $I(z'p'p'')$. From $p' \neq p''$, $I(s'p'p'')$,

$I(y'p'p'')$, $I(z'p'p'')$ we deduce that $L(s'y'z')$ (by (3.44)), which in turn implies that $L(x'y'z')$ (by A3.3.6, A3.2.2, A3.2.3 and the fact that $s' \neq y'$ (by (3.11), since $x \neq y$, therefore $x' \neq y'$ (by (3.35))), which implies $L(xyz)$ (by (3.33), (3.42)). This contradicts our hypothesis, hence $p' = p''$, i. e. (3.61) holds with $\neg L(xyz)$ added to its antecedent. If $L(xyz)$, then (3.61) follows from A3.3.7, (3.43) and A3.2.6.

Let $a \neq b$, $R(abx) = x'$, $R(aby) = y'$, $R(abu) = u'$, $R(abv) = v'$ and $xy \| uv$, i. e. $x \neq y$, $u \neq v$ (hence $x' \neq y'$ and $u' \neq v'$ as well by (3.35)) and $L(uvP(yxu))$ (by (3.59)). By (3.61) and (3.42) we conclude that $L(u'v'P(y'x'u'))$, i. e. $x'y' \| u'v'$ (by (3.59)). This proves A3.4.21.

A3.4.22 is (3.33) and a proof of A3.4.23 from A3.2.1-A3.2.9, A3.4.13, A1.1.1, A1.1.3, A3.2.11 was given in [77, (1.15)].

We have thus proved the following

Representation Theorem 3.5.1 *$\mathfrak{M} \in Mod(\mathcal{CRE})$ iff $\mathfrak{M} \simeq \langle E, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{P}_K, \mathbf{R}_{(K,k)} \rangle$, with E and $\mathbf{P}_K$ as in the Representation Theorem 3.4.2, $\mathbf{a}_0 = (0,0)$, $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (\alpha, \beta)$, with α, β in K, $\beta \neq 0$, and with $\mathbf{R}_{(K,k)}$ same as $\mathbf{R}'_{(K,k)}$ whenever the first two arguments are different, and $\mathbf{R}_{(K,k)}(\mathbf{aab}) = \mathbf{P}_K(\mathbf{baa})$.*

$\mathfrak{M} \in Mod(\mathcal{CRE}')$ iff $\mathfrak{M} \simeq \langle E, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{P}_K, \mathbf{R}_{(K,1)} \rangle$, with E, $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{P}_K, \mathbf{R}_{(K,1)}$ as above (with $k = 1$), with $\alpha = 0$, $\beta = 1$.

Corollary 3.5.1 (to Theorem 3.5.1 and Theorem 3.3.1) *$\mathcal{CRE} \subset Cn(\mathcal{CME} \cup \{(3.2)\})$.*

3.6 Independence of the language primitives

We now ask the natural question whether both operations P and F are needed for a constructive (i. e. quantifier-free) axiomatization of $\mathcal{CME}$. Since P is an affine operation, F is needed (as we could not ensure that conditions (ii) and (iii) in § 3.2 would be satisfied, if we had only P at our disposal), but we were not able to prove that P is needed. The same applies to the question whether both P and R are needed to constructively axiomatize $\mathcal{CRE}$: R is definitely needed, but we do not know whether P is needed as well.

We conjecture that the answer is affirmative in both cases.

3.7 Simplicity revisited

> There are only four conditions, namely, primary condition,
> objectively supporting condition, immediately contiguous
> condition, and dominant condition.
> A fifth condition does not exist.
>
> Nāgārjuna, *Mūlamadhyamakakārikā.*

In § 2.2 we provided an axiom system for $\mathcal{E}'_2$ (the $\mathrm{L}_{B\equiv}$-theory of Cartesian planes over Euclidean ordered fields (cf. also [51])), all of whose axioms have, when written in prenex form, at most 5 variables. It was also proved that this result cannot be improved, in the sense that there is no axiom system for $\mathcal{E}'_2$ all of whose axioms have at most 4 variables (cf. § 2.2.3), i. e. that $sd(\mathcal{E}'_2) = 5$.

According to D. SCOTT's theorem (cf. § 2.3), any axiom system for plane Euclidean geometry (by which we mean any theory in $[\mathcal{E}_2]$, $[\mathcal{E}_2^-]$, $[\mathcal{E}_2^{-q}]$, $[\mathcal{BD}_2]$, $[\mathcal{BD}_2']$, $[\mathcal{D}_2]$, $[\mathcal{D}_2']$, $[\mathcal{ME}]$, $[\mathcal{ME}']$, $[\mathcal{RE}]$, $[\mathcal{RE}']$, as well as $[\mathcal{E}_2']'$) must have some axiom, that, when written in prenex form, has at least 4 variables.

We now want to prove that

Theorem 3.7.1
$Sd([\mathcal{T}]) = 4$, where $\mathcal{T}$ is any of $\mathcal{E}_2$, $\mathcal{E}_2^-$, $\mathcal{E}_2^{-q}$, $\mathcal{BD}_2$, $\mathcal{BD}_2'$, $\mathcal{D}_2$, $\mathcal{D}_2'$, $\mathcal{ME}$, $\mathcal{ME}'$, $\mathcal{RE}$, $\mathcal{RE}'$, and $Sd([\mathcal{E}_2']') = 4$.

3.7.1 The Simplicity degree of $[\mathcal{ME}]$ and of $[\mathcal{ME}']$ is 4

Among the axioms we have proposed for $\mathcal{CME}$ (i. e. A3.2.1, A3.3.1-A3.3.11), the only one that requires more than 4 variables is A3.3.4. We shall prove that A3.3.4 may be replaced by A3.7.1 and A3.7.2 , where

A 3.7.1 $a \neq c \wedge a \neq b \wedge F(abc) = c \rightarrow F(abx) = F(acx)$,

A 3.7.2 $F(abx) = F(bax)$.

Suppose that $a \neq b \wedge c \neq d \wedge F(abc) = c \wedge F(abd) = d$. If $c = a$, then $F(abx) = F(cdx)$ (by A3.7.1); if $d = a$, then the same conclusion is reached using A3.7.1 and A3.7.2. If $c \neq a$ and $d \neq a$, then $F(abx) = F(acx)$, $F(abx) = F(adx)$ (by A3.7.1), hence $F(acx) = F(adx)$. In particular, with $x = c$ and using A3.7.2, $F(cac) = F(dac)$, which implies $F(dac) = c$ (since $F(cac) = c$ (by A3.2.1)). We conclude that $F(dax) = F(dcx)$ (by A3.7.1), which, together with $F(abx) = F(adx)$ (by A3.7.1), gives $F(abx) = F(cdx)$ (by A3.7.2) and proves A3.3.4.

The axiom system $\Omega(F, P) = \{$A3.2.1, A3.3.1-A3.3.3, A3.7.1, A3.7.2, A3.3.5-A3.3.11$\}$ is simple, regardless of language, for $[\mathcal{ME}]$, and $Sd([\mathcal{ME}]) = 4$. The result extends, of course, to $\mathcal{CME}'$ as well.

3.7.2 The Simplicity degree of $[\mathcal{RE}]$ and of $[\mathcal{RE}']$ is 4

The above result also extends to $\mathcal{CRE}$ and $\mathcal{CRE}'$, since, for their axiom systems, it is again only axiom A3.5.1 that requires more than 4 variables, and, just as with A3.3.4, it may be replaced by A3.7.3 and A3.7.4 , where

A 3.7.3 $a \neq c \wedge a \neq b \wedge R(abc) = c \rightarrow R(abx) = R(acx)$,

A 3.7.4 $R(abx) = R(bax)$.

Therefore the axiom system $\Omega(R, P) =$ (A3.2.1, A3.3.1-A3.3.3, A3.7.3, A3.7.4 A3.3.5-A3.3.11, (3.19)), where the axioms are considered to be L_{re}-sentences is simple, regardless of language, for $[\mathcal{RE}]$, and $Sd([\mathcal{RE}]) = 4$. This easily extends to $[\mathcal{RE}']$.

3.7.3 The Simplicity degree of $[\mathcal{D}_2]$ and of $[\mathcal{D}_2']$ is 4

The only statement we need in order to get from either metric-Euclidean planes or rectangular planes to Euclidean planes is the Eucidean parallel axiom, which we adopt in the form A2.2.9, since in this form it can be constructively expressed by using an additional ternary predicate U, having the intuitive interpretation

$$U(abc) = d \quad \text{iff} \quad \begin{array}{l} d \text{ is the centre of the circumcircle of } \triangle abc, \\ \text{provided that } a,\ b,\ c \text{ are three non-collinear points,} \\ \text{arbitrary, otherwise.} \end{array}$$

In this constructive language A2.2.9 takes the form

A 3.7.5 $\neg L(abc) \rightarrow I(U(abc)ab) \wedge I(U(abc)bc)$

In order to prove that $\Omega(F, P) \cup \{\text{A3.7.5}\}$ and $\Omega(R, P) \cup \{\text{A3.7.5}\}$ are indeed axiom systems for theories in $[\mathcal{D}_2]$, we only need to show that $\Omega_1 = \Omega(R, P) \cup \{\text{A3.7.5, (3.1), (3.57), (3.58), (3.5)}\} \vdash \mathcal{D}_2$, since $\Omega_0 = \Omega(F, P) \cup \{\text{A3.7.5, (3.1), (3.2), (3.3), (3.4), (3.5)}\} \vdash \mathcal{D}_2$ will then folow, as $\Omega_1 \subset Cn(\Omega_0)$ (by Corollary 3.5.1).

Since $\mathcal{D}_2 = Cn(\{\text{A2.1.1 - A2.1.4, A1.1.1, A2.1.13}\})$ (cf. §2.1) and A2.1.1, A2.1.2, A1.1.1 have been shown to be true both in Ω_1, A2.1.4 clearly follows from A3.7.5, (3.57), (3.58), (3.5), and A2.1.13 is clearly true in Ω_1 (it is certainly true that there is a triangle and a midpoint), all we need to prove is that A2.1.3 holds in Ω_1.

To prove that the existence statement in A2.1.3 holds in Ω_1, let $\triangle abc$ and $\triangle a'b'c'$ be two congruent triangles (although the hypothesis assumes only $\triangle abc$ to be a proper triangle (i. e. $\neg L(abc)$), one can derive from (3.41) and the fact that $\equiv$ is an equivalence relation, that $\triangle a'b'c'$ is also proper). If $P(aa'b) = b'$ and $P(aa'c) = c'$, then the point x' we are looking for will be $P(aa'x)$. If $P(aa'b) = b'$, but $P(aa'c) \neq c$, then, since $\triangle a'b'c'$ and $\triangle a'b'P(aa'c)$ are congruent (since translations are isometries), we must have $c' = R(a'b'P(aa'c))$ (by A3.3.5) and we let $x' = R(a'b'P(aa'x)$. If $P(aa'b) \neq b'$, then $\triangle a'b'c'$ and $\triangle a'b'R(a'R(b'P(aa'b)a')P(aa'c))$ are congruent (since both translations and reflections are isometries), hence either $R(a'R(b'P(aa'b)a')P(aa'c)) = c'$ and then we let $x' = R(a'R(b'P(aa'b)a')P(aa'x))$, or $R(a'R(b'P(aa'b)a')P(aa'c)) \neq c'$, i. e. $R(a'b'R(a'R(b'P(aa'b)a')P(aa'c))) = c'$ (by A3.3.4) and then we let $x' = R(a'b'R(a'R(b'P(aa'b)a')P(aa'x)))$.

The uniqueness statement follows from (3.44) and the fact that $\equiv$ is an equivalence relation.

A proof that metric-Euclidean planes that satisfy the parallel axiom are Euclidean planes can also be found in [10, §13].

3.7.4 Constructive Euclidean planes can be axiomatized in terms of R' and U

We now want to prove that the operation P becomes redundant once we have an operation for reflection in lines and U, so that we may axiomatize by quantifier-free axioms constructive Euclidean planes in a language $\mathrm{L}_{eu} = \mathrm{L}(a_0, a_1, a_2, R', U)$ with only a_0, a_1, a_2, R' and U as primitive notions, where R' is a ternary operation, such that $R'(abc)$ has the same interpretation as $R(abc)$ for $a \neq b$, but takes arbitrary values for $a = b$. To put it more simply, R stands for the operation of reflection in both lines and points, whereas R' stands for reflection in lines only. To constructively define P and σ from U and R', it is enough to show that M and σ can be constructively defined from U and R, M being a binary operation with the intuitive meaning '$M(ab)$ is the midpoint of the segment ab'.

The abbreviation L will be now introduced as
$L(abc) \overset{\text{def}}{\leftrightarrow} R'(abc) = c \vee a = b$.
For the next abbreviations u, α, β, γ, δ, ϵ, ζ see Fig. 3.2, for φ see Fig. 3.1.
$u_n = u_n(x, y) \overset{\text{def}}{=} U(a_nxy)$,
$\alpha = \alpha(x, y, q) \overset{\text{def}}{=} R'(qR'(xyq)y)$,
$\beta = \beta(x, y, p, q) \overset{\text{def}}{=} R'(pR'(xyp)q)$,
$\gamma = \gamma(x, y, p, q) \overset{\text{def}}{=} R'(xy\beta)$,
$\delta = \delta(x, y, p, q) \overset{\text{def}}{=} R'(\beta\gamma y)$,
$\epsilon = \epsilon(x, y, p, q) \overset{\text{def}}{=} R'(p\alpha\delta)$,
$\zeta = \zeta(x, y, p, q) \overset{\text{def}}{=} U(\delta\alpha\epsilon)$, $\varphi(x, y, p, q) = z \overset{\text{def}}{\leftrightarrow} \neg L(xyp) \wedge R'(pR'(xyp)y) = x$
$\wedge \neg L(xyq) \wedge R'(qR'(xyq)y) \neq x \wedge U(q\beta\gamma) = z$,

$$\begin{aligned} M(xy) = z \quad \overset{\text{def}}{\leftrightarrow} \quad & (x = y \wedge y = z) \vee (\bigvee_{n=0}^{2} (\neg L(xya_n) \wedge ((L(u_nxy) \wedge u_n = z) \\ & \vee \varphi(x, y, a_n, U(R'(a_nxy)xa_n)) = z \vee \varphi(x, y, u_n, a_n) = z))). \end{aligned} \tag{3.62}$$

This definition of M in terms of U and R' 'says':
'See if $\angle xa_ny$ is a right angle for some a_n; if so let $z = U(a_nxy)$.
If $\angle xa_ny$ is not right, then find a point p on the perpendicular bisector of the segment xy (different from the midpoint itself) and another point q not on it and not on the line xy; do $\varphi(x, y, p, q)$ (see Fig. 3.1), i. e. find β , γ and then z as $U(q\beta\gamma)$'.
To be precise: (for $x \neq y$) 'Choose an a_n such that $\neg L(xya_n)$ (this will be possible since, for $x \neq y$, if we had $L(xya_0)\wedge L(xya_1)\wedge L(xya_2)$, then, by A3.2.1-A3.2.3 (which were proved without using any axiom containing P), we should have $L(a_0a_1a_2)$, which contradicts A3.3.11); the choice of p and q will depend on the position of a_n relative to x and y.
If a_n lies on the perpendicular bisector of the segment xy, then let $p = a_n$ and $q = U(R'(a_nxy)xa_n))$; if a_n does not lie on the perpendicular bisector of xy, then let $q = a_n$ and $p = U(a_nxy)$'.
$\psi(x, y, p, q) = z \leftrightarrow \neg L(pxy)\wedge R'(pR'(xyp)y) = x\wedge\neg L(xyq)\wedge R'(qR'(xyq)y) \neq x\wedge R'(x\zeta y) = z$ (see Fig. 3.2),
$\sigma(xy) = z \leftrightarrow (x = y \wedge y = z) \vee (\bigvee_{n=0}^{2}(\neg L(xya_n) \wedge (\psi(x, y, a_n, U(R'(a_nxy)xa_n)) = z \vee \psi(x, y, U(R'(a_nxy)xy), a_n) = z)))$. This definition of σ 'says':
'Find a point p on the perpendicular bisector of the segment xy (different from the midpoint itself) and another point q not on it; do $\psi(x, y, p, q)$ (see Fig. 3.2), i. e. find $R'(xyq)$, α, $R'(xyp)$, β, γ, δ, ϵ (ϵ was chosen such as to make sure that $\zeta \neq x$), ζ and then z as $R'(x\zeta y)$ (note that $\zeta x \perp xy$)'.
To be precise: (for $x \neq y$) 'Choose an a_n such that $\neg L(xya_n)$; the choice of p and q will de-

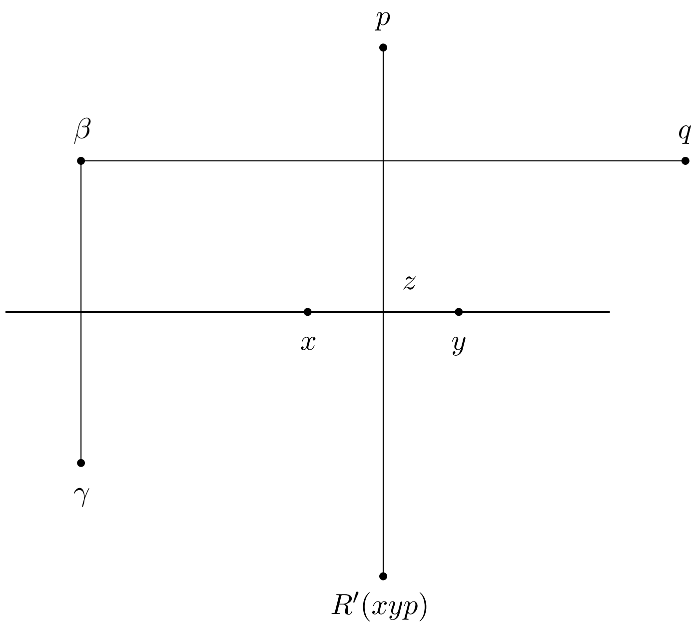

Figure 3.1: The definition of $M(xy)$

pend on the position of a_n relative to x and y. If a_n lies on the perpendicular bisector of the segment xy, then let $p = a_n$ and $q = U(R'(a_nxy)xa_n)$; if a_n does not lie on the perpendicular bisector of xy, then let $q = a_n$ and $p = U(R'(a_nxy)xy)$'.

We are finally ready to define P by

$$P(xyz) \overset{\text{def}}{\leftrightarrow} \sigma(M(yz)x). \tag{3.63}$$

Since R may also be defined by

$$R(abc) = d \overset{\text{def}}{\leftrightarrow} (a \neq b \wedge R'(abc) = d) \vee (a = b \wedge \sigma(ac) = d), \tag{3.64}$$

we can restate the axiom system for $\mathcal{CRE}$ in L_{eu}, without introducing any new variable or quantifier. This will, however multiply the axioms, since each axiom has to be stated in each of the particular cases that were involved in the definitions of M and σ. For example, A3.3.6, which was $L(xy\sigma(xy))$, will have to be replaced by the 6 statements
$\theta(n,p,q):\ [x = y \vee (\neg L(xya_n) \wedge \neg L(pxy) \wedge R'(pR'(xyp)y) = y \wedge \neg L(xyq)$
$\wedge\, R'(qR'(xyq)y) \neq x)] \rightarrow L(xyR'(x\zeta(x,y,p,q)y)$,
for $n = 0,1,2$, $(p,q) = (U(R(a_nxy)xy), a_n), (a_n, U(R(a_nxy)xa_n))$ (where L and $\zeta(x,y,p,q)$ will have to be replaced by their respective definitions; they do not require a case distinction). We have thus proved that $Sd([\mathcal{D}_2]) = Sd([\mathcal{D}_2']) = 4$.

Let $\mathcal{CD}_2 = Cn_{\mathrm{L}_{eu}}(\Omega(R,P) \setminus \{(3.19)\}, \mathrm{A}3.7.5)$ and $\mathcal{CD}_2' = Cn_{\mathrm{L}_{eu}}(\mathcal{CD}_2, \mathrm{A}3.5.2)$ (we want to emphasize that the axioms for $\mathcal{CD}_2$ and $\mathcal{CD}_2'$ are considered here in their conversions to L_{eu}-axioms). From Representation Theorems 2.1.2 and 2.1.1(iv) we get

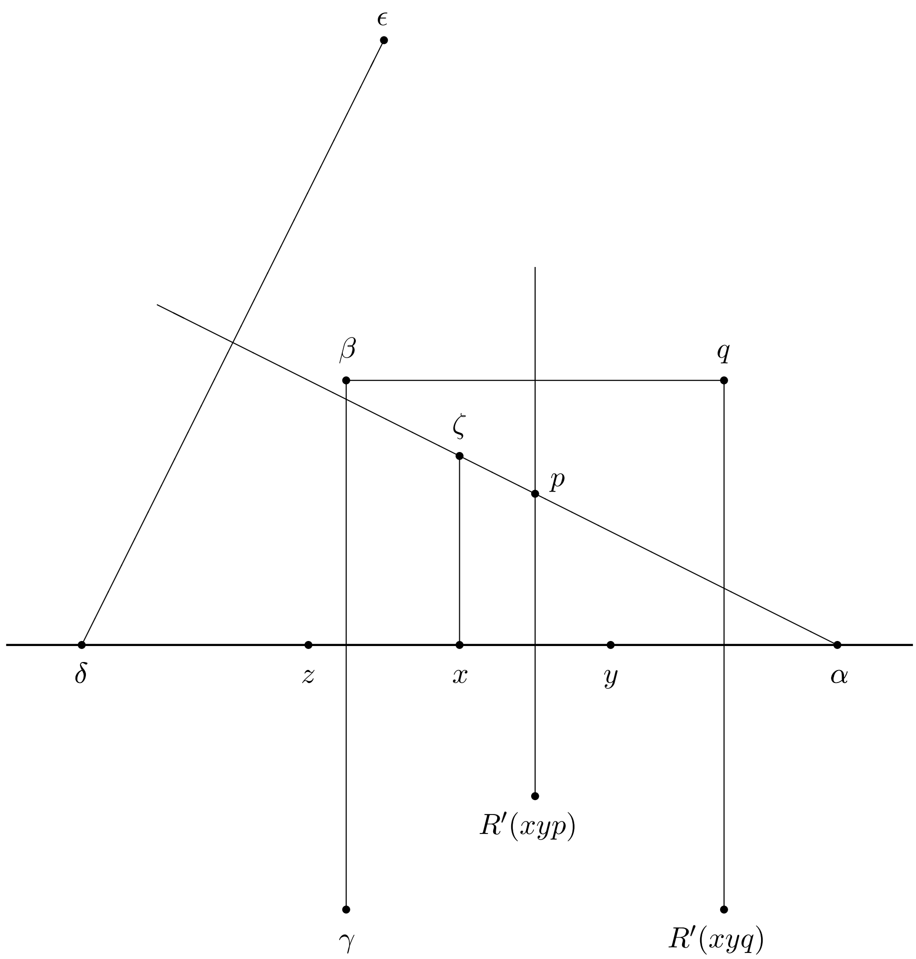

Figure 3.2: The definition of $\sigma(xy)$

Representation Theorem 3.7.1 *$\mathfrak{M} \in Mod(\mathcal{CD}_2)$ iff $\mathfrak{M} \simeq \mathfrak{E}_2(F,k)$, where F is a field, $k \in F$, $-k \notin F_2$, and $\mathfrak{E}_2(F,k) = \langle F \times F, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{R}'_{(F,k)}, \mathbf{U}_{(F,k)}\rangle$, with*
$\mathbf{R}'_{(F,k)}(\mathbf{xyz}) = \mathbf{t} = (t_1, t_2)$, if $\mathbf{x} \neq \mathbf{y}$, with
$t_1 = (k(2x_1 - z_1)(y_2 - x_2)^2 + z_1(y_1 - x_1)^2 + 2k(y_2 - x_2)(z_2 - x_2)(y_1 - x_1))(\|\mathbf{y} - \mathbf{x}\|)^{-1}$,
$t_2 = ((2x_2 - z_2)(y_1 - x_1)^2 + kz_2(y_2 - x_2)^2 + 2(y_1 - x_1)(z_1 - x_1)(y_2 - x_2))(\|\mathbf{y} - \mathbf{x}\|)^{-1}$,
arbitrary, otherwise;
$\mathbf{U}_{(F,k)}(\mathbf{xyz}) = (C^{-1}(A(y_2 - x_2) - B(z_2 - x_2)), (Ck)^{-1}(B(z_1 - x_1) - A(y_1 - x_1)))$,
with $A = \|\mathbf{z}\| - \|\mathbf{x}\|$, $B = \|\mathbf{y}\| - \|\mathbf{x}\|$, $C = 2((z_1 - x_1)(y_2 - x_2) - (y_1 - x_1)(z_2 - x_2))$, if $C \neq 0$;
arbitrary, otherwise;
$\mathbf{a}_0 = (0,0), \mathbf{a}_1 = (1,0), \mathbf{a}_2 = (\alpha, \beta)$, with α, $\beta \in F$, $\beta \neq 0$.

$\mathfrak{M} \in Mod(\mathcal{CD}'_2)$ iff $\mathfrak{M} \simeq \mathfrak{E}_2(F,1)$, where F is a 2-formally real field, and $(\alpha, \beta) = (0,1)$.

We ask again the natural question: Are both operations R' and U needed in order to axiomatize $\mathcal{CD}_2$ (or $\mathcal{CD}'_2$) by quantifier-free axioms? We shall prove (in much the same way as was done in [8, p. 84] or in [39, p. 107-109]) that U is indeed needed, i.e. that there is no universal axiom system Δ in $\mathrm{L}(a_0, a_1, a_2, R')$, such that $\mathfrak{M} \in Mod(\Delta)$ iff $\mathfrak{M} \simeq \langle F \times F, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{R}'_{(F,1)}\rangle$, where F is a 2-formally real field. If there were such an axiom system, then, given $\mathbf{a}_0 = (x_0, y_0)$, $\mathbf{a}_1 = (x_1, y_1)$, $\mathbf{a}_2 = (x_2, y_2)$ in $F \times F$ satisfying A3.3.11, the smallest set containing $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ and closed under the operation $\mathbf{R}'$ would have to be $F' \times F'$, where $F' = P(x_0, x_1, x_2, y_0, y_1, y_2)$ and P is the prime field of F.

Let $F = \mathbb{Q}(t)$, ordered by $(\sum_{i=0}^{n} a_i t^i)(\sum_{j=0}^{m} a_j t^j)^{-1} > 0$ iff $a_n b_m > 0$, let $\mathbf{a}_0 = (0,0), \mathbf{a}_1 = (t,0), \mathbf{a}_2 = (0,t)$ and let $S = \{(x,y) \in F \times F \mid |x| < nt, |y| < nt, \text{ for some } n \in \mathbb{N}\}$. We shall say that an n-ary operation $\mathbf{O}$ is *local* if S is closed under $\mathbf{O}$, i. e. if $\mathbf{O}(\mathbf{x}_1 \ldots \mathbf{x}_n) \in S$ whenever $\mathbf{x}_1 \ldots \mathbf{x}_n$ are in S. If there were a quantifier-free axiom system for plane geometry (over any class of fields) in $\mathrm{L}(a_0, a_1, a_2, O_1, \ldots O_r)$, where the O_i's are operation symbols, then $Cl(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, the closure of $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ under the operations $\mathbf{O}_1, \ldots \mathbf{O}_r$, would have to include $F \times F$. If all the $\mathbf{O}_i$'s are local, then $Cl(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ would have to be included in S, since $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2 \in S$; so we should have $F \times F \subseteq Cl(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) \subseteq S$, a contradiction, since $(0, t^2) \in F \times F \setminus S$. Hence no finite set of local operation symbols and individual constants can axiomatize Euclidean planes (over any class of fields). Since R' is a local operation, we are done.

We conjecture that R' is needed as well, i. e. that there is no universal axiom system for Euclidean planes that uses only the operation U (in fact, we believe that there is no axiom system in first-order logic that uses only the operation U, because we conjecture that L cannot be defined in terms of U).

A different axiom system for $\mathcal{CD}_2$ (and $\mathcal{CD}'_2$), some of whose axioms require more than 4 variables, was given in [54].

3.7.5 The Simplicity degree of $\mathcal{BD}_2$ and of $\mathcal{BD}'_2$ is 4

W. Szmielew [87] has shown that in order to get from $\mathcal{D}_2$ to $\mathcal{BD}_2$ one needs to add to $\mathcal{D}_2$ the axioms A2.1.7, A2.1.8, A1.1.9 and A2.1.9 (her proof shows that these are the axioms one needs to add in order to get from $\mathcal{E}_2 \cap \mathrm{L}_D$ to $\mathcal{E}_2$, but it remains unchanged in our case; an alternate proof that directly applies to our case, since it is carried out for an affine space and

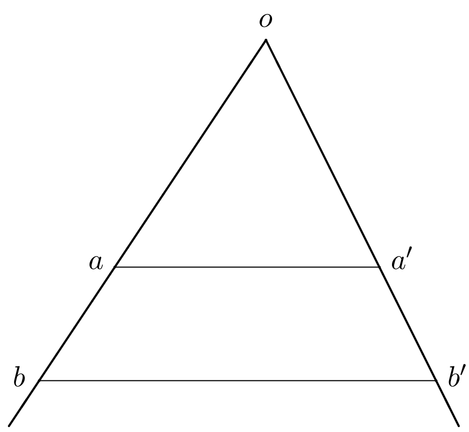

Figure 3.3: The minor PASCH axiom

$\mathcal{D}_2$ has an underlying affine structure, can be found in [88, Theorem 8. 1. 7] and [60, Satz V. 1. 4]).

A2.1.9, the weak PASCH axiom, may be reformulated as (see Fig. 3.3)

A 3.7.6 $o \neq a \wedge a \neq b \wedge B(oab) \wedge L(oa'b') \wedge aa' \| bb' \rightarrow B(oa'b')$

where the parallelity relation $\|$ is defined as usual (cf. e. g. [88]) in terms of L, which in turn is defined by (1.1). In this form it is called *kleines Axiom von* PASCH in [60]. It is easy to see that A3.7.6 may be replaced by that special case of it, in which we add to the antecedent the requirement that aa' and bb' be perpendicular to the line oa'. In the language L_{me}, enlarged by the ternary relation B, it becomes

A 3.7.7 $o \neq a \wedge a \neq b \wedge B(oab) \wedge o \neq o' \rightarrow B(oF(oo'a)F(oo'b))$.

Since both A2.1.7 and A2.1.8 require at most 4 variables, and A1.1.9 may be replaced by (1.1) (where L is considered to be defined by (3.57)), which requires only 3 variables, we conclude that the axiom system for $\mathcal{CD}_2$ (respectively $\mathcal{CD}'_2$) given in § 3.7.3 to which we add A2.1.7, A2.1.8, (1.1) and A3.7.7 is simple with simplicity degree 4. This proves that $Sd([\mathcal{BD}_2]) = Sd([\mathcal{BD}'_2]) = 4$. The language in which it is expressed, i. e. $\mathrm{L}(a_0, a_1, a_2, R', F, U, B)$ contains the redundant symbol F and is not constructive, since it contains a predicate symbol. This can be avoided by introducing an operation O_B instead of B, which is its decision operation, i. e.

$$O_B(xyz) = \begin{cases} x & \text{if } B(zxy) \\ y & \text{if } B(xyz) \\ z & \text{if } B(yzx) \\ \mathit{arbitrary} & \text{if } \neg L(xyz) \end{cases}$$

We may then replace every occurrence of $B(xyz)$ by $(x = y \vee R'(xyz) = z) \wedge O_B(xyz) = y$ (and F by its definition in terms of a_0, a_1, a_2, R' and U — cf. [54] for a definition of M in terms of these; for $a \neq b$ we have $F(abc) = M(cR'(abc))$) and obtain a constructive axiom system for a theory in $[\mathcal{BD}_2]$ (respectively $[\mathcal{BD}'_2]$), expressed in the language $\mathrm{L}_{oe} = \mathrm{L}(a_0, a_1, a_2, R', U, O_B)$, that we denote by $\mathcal{CBD}_2$ (respectively $\mathcal{CBD}'_2$).

The operation O_B can obviously not be defined by R' and U and we conjecture that these operations are constructively independent, in the sense that one cannot axiomatize $\mathcal{CBD}_2$ (respectively $\mathcal{CBD}'_2$) using only two of these operations.

3.7.6 The Simplicity degree of $[\mathcal{E}_2]$, $[\mathcal{E}_2^-]$ and of $[\mathcal{E}_2^{-q}]$ is 4

In order to get from $\mathcal{BD}_2$ to $\mathcal{E}_2$ we need a transport axiom, for the fact that the coordinate field must be a Pythagorean ordered field means that we have, geometrically speaking, *free mobility*. We thus have to introduce a new ternary operation T, with the intuitive meaning

$T(abc) = d$ iff the point d is as distant from a on the ray $\overrightarrow{ac}$ as b is from a,
provided that $a \neq c \vee (a = c \wedge a = b)$,
arbitrary, otherwise.

We can define B by means of T through

$$B(abc) \leftrightarrow (a \neq c \wedge T(abc) = b \wedge T(cba) = b) \vee (a = c \wedge a = b), \tag{3.65}$$

and we need only one axiom to describe how T operates metrically, namely

A 3.7.8 $c \neq a \vee a = b \rightarrow ab \equiv aT(abc) \wedge (B(acT(abc)) \vee B(aT(abc)c))$.

Therefore, if we add to the axiom system for $\mathcal{CD}_2$ given in § 3.7.4 the axioms A2.1.8, (1.1), A3.7.7 and A3.7.8[2], we get a simple axiom system for a theory in $[\mathcal{E}_2]$ with simplicity degree 4. This proves that $Sd([\mathcal{E}_2]) = 4$. This axiom system would, however, be expressed in a language with the operation symbols R', U, T as primitive notions (beside the three individual constants). We shall prove that R' is superfluous, i. e. that we can axiomatize constructive plane geometry over Pythagorean ordered fields using only the operations U and T.

W. Szmielew [87] has shown that to get from $\mathcal{E}_2 \cap \mathrm{L}_D$ to $\mathcal{E}_2^-$, we need the axioms A1.1.9, A2.1.7, A2.1.8, A3.7.9, where A3.7.9 is a particular instance of the axiom that states that isometries preserve the order relation on a line, whose exact $\mathrm{L}_{B\equiv}$ statement is

A 3.7.9 $ab \equiv ab' \wedge bc \equiv b'c' \wedge ac \equiv ac' \wedge B(abc) \rightarrow B(ab'c')$.

Expressed in the constructive language $\mathrm{L}(a_0, a_1, a_2, R', U, T)$, it becomes

A 3.7.10 $o \neq a \wedge a \neq b \wedge B(oab) \wedge o \neq o' \rightarrow B(oR'(oo'a)R'(oo'b))$.

Therefore, if we add to the axiom system for $\mathcal{CD}_2$ given in§ 3.7.4 the axioms A2.1.8, (1.1), A3.7.10 and A3.7.8, we get an axiom system for constructive semi-ordered Euclidean planes with free mobility, i. e. the coordinate field is a semi-ordered Pythagorean field, U, and R have their standard interpretations, and T is interpreted as in Representation Theorem 3.7.2 (see §3.7.7), with the $\sqrt{x}$ being interpreted as the semi-positive square root, i. e. the one that belongs to the semi-positive cone. This proves that $Sd([\mathcal{E}_2^-]) = 4$.

If we also add an axiom stating that the hypotenuse is greater than the sides of a right triangle, i. e.

[2] Notice that A2.1.7 is a consequence of (3.65).

A 3.7.11 $a \neq b \wedge c \neq a \wedge F(abc) = a \rightarrow B(caT(cba))$,

we get an axiom system for constructive quadratically semi-ordered Euclidean planes with free mobility (i. e. the coordinate field is a quadratically semi-ordered Pythagorean field). This proves that $Sd([\mathcal{E}_2^{-q}]) = 4$.

3.7.7 Constructive ordered Euclidean planes with free mobility can be axiomatized in terms of U and T

We shall first define σ and M using T and U. We have
$u_n = u_n(x,y) \overset{\text{def}}{=} U(a_nxy)$,
$\varphi(x,y,p) = z \overset{\text{def}}{\leftrightarrow} B(pxy) \wedge p \neq x \wedge T(xyp) = z$,

$$\begin{aligned}\sigma(xy) = z \ \overset{\text{def}}{\leftrightarrow}\ & (x = y \wedge y = z)\\ & \vee(\bigvee_{n=0}^{2}(\neg L(xya_n) \wedge [\neg B(xu_ny) \wedge (\varphi(x,y,T(T(yu_nx)yx)) = z\\ & \vee\varphi(x,y,T(yu_nx)) = z \vee (\varphi(x,y,T(T(yU(u_nxy)x)yx)) = z)]\\ & \vee[B(xu_ny) \wedge (\varphi(x,y,T(T(u_nT(xa_ny)x)yx)) = z\\ & \vee\varphi(x,y,T(T(ya_nx)yx)) = z)])),\end{aligned}$$

$$\begin{aligned}M(xy) = z \ \overset{\text{def}}{\leftrightarrow}\ & (x = y \wedge y = z)\\ & \vee(\bigvee_{n=0}^{2}(\neg L(xya_n) \wedge [\neg B(xu_ny) \wedge (u_n = z\\ & \vee U(U(T(yx\sigma(u_nx))xy)xy) = z])).\end{aligned}$$

The definition of $\sigma(xy) = z$ in terms of T and U ‘says’ (see Fig. 3.4):
Find a point p ($p \neq x$) such that x lies between p and y, and let z be $T(xyp)$.

To be precise: If $x = y$ then let z be x. If $x \neq y$, then choose an a_n such that $\neg L(xya_n)$ (there is such an a_n because $\neg L(a_0a_1a_2)$); denote by u the centre of the circumcircle of $\triangle a_nxy$, i. e. $u_n = U(a_nxy)$. The choice of p depends on the position of a_n with respect to x and y.

If $\angle xa_ny$ is not a right angle, then denote by d the point on the ray yx such that $|yd| = |yu_n|$, i. e. $d = T(yu_nx)$;

(i) if d lies between x and y, then take $p = T(dxy)$;

(ii) if x lies between d and y, then take $p = d$;

(iii) if $d = x$, i. e. if $\triangle uxy$ is an equilateral triangle, then let $u_n' = U(xu_ny)$ and $d' = T(yu_n'x)$; d' lies between x and y, so, just as in (i), take $p = T(d'yx)$.

If $\angle xa_ny$ is a right angle, then fix the point p as follows:

(iv) if $|xa_n|$ is greater than $|xy|/2$, then let t be the point on the ray $\overrightarrow{xy}$ such that $|xt| = |xa_n|$, and let t' be the point that is symmetric to t with respect to u, obtained as $T(u_ntx)$, and take p to be the point that is symmetric to y with respect to y', obtained as $T(t'yx)$;

(v) if $|ya_n|$ is greater than $|xy|/2$, then let $d = T(ya_nx)$, and, just as in (i), take $p = T(dyx)$.

The definition of M in terms of U and T 'says' (see Fig. 3.5): If $x = y$, then take $z = y$. If $x \neq y$, then choose an a_n such that $\neg L(xya_n)$, and let $u_n = U(a_nxy)$. If $\angle xa_ny$ is a right angle, then take $z = u_n$; otherwise let b be the symmetric point of x with respect to u_n, i. e. $b = \sigma(u_nx)$; let d be the point on the ray $\overrightarrow{yb}$ such that $|xy| = |yd|$ and let $e = U(xyd)$; then e is the midpoint of the segment dx and $\angle xey$ is a right angle ; take $z = U(xey)$.

With $m \stackrel{\text{def}}{=} M(T(yzx)z)$, we are now ready to define R' in terms of U and T:

$$\begin{aligned} x \neq y \wedge R'(xyz) = t \;\; \stackrel{\text{def}}{\leftrightarrow} \;\; & x \neq y \wedge (L(xyz) \wedge t = z) \\ & \vee \sigma(T(y(m\sigma(T(y(mx)T(yzm)))T(yzx)) = t). \end{aligned} \tag{3.66}$$

The definition of R' 'says' (see Fig. 3.6): If z lies on the line xy, then take $t = z$; otherwise, let $b = T(yzx)$, $d = M(bz)$, $e = T(ydx)$, $f = T(yzd)$, $g = \sigma(ef)$ and $h = T(ydg)$, and take $t = \sigma(hb)$ to be the reflection of z in the line xy. We can now restate the axioms for the constructive theory of Cartesian planes over Pythagorean ordered fields in $\mathrm{L}_e = \mathrm{L}(a_0, a_1, a_2, T, U)$ without adding any additional variable or quantifier. The resulting axiom system is both *constructively expressed in the simplest possible language* (since T alone, as a *local* operation[3], cannot constructively axiomatize constructive Cartesian planes, and U alone would not be able to produce the square roots of sums of squares (so if the $\mathbf{a}_i$'s are all rational, then their closure under U is included in $\mathbb{Q} \times \mathbb{Q}$)) and *simple, regardless of language*, so it is simple according to the most stringent syntactical simplicity criteria 1.2.3 and 1.2.4.[4]

Let $\mathcal{CE}_2 \stackrel{\text{def}}{=} Cn_{\mathrm{L}_e}(\mathcal{CD}_2, \text{A2.1.8, A3.7.7, A3.7.8})$. We have shown that

Representation Theorem 3.7.2 *$\mathfrak{M} \in Mod(\mathcal{CE}_2)$ iff $\mathfrak{M} \simeq \mathfrak{P}_2(F)$, where F is an ordered Pythagorean field and $\mathfrak{P}_2(F) = \langle F \times F, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{T}_F, \mathbf{U}_F \rangle$, with $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{U}_F = \mathbf{U}_{(F,1)}$ defined as in Representation Theorem 3.7.1, $\|\mathbf{x}\| = x_1^2 + x_2^2$, and*

$$\mathbf{T}_F(\mathbf{xyz}) = \begin{cases} (x_1 - AB^{-1}(x_1 - z_1), x_2 - AB^{-1}(x_2 - z_2)), \\ \textit{with } A = \sqrt{\|\mathbf{x}-\mathbf{y}\|}, B = \sqrt{\|\mathbf{x}-\mathbf{z}\|} \textit{ if } B \neq 0, \\ \textit{arbitrary, otherwise.} \end{cases}$$

A different axiom system for $\mathcal{CE}_2$, some of whose axioms require more than 4 variables, was given in [51].

3.7.8 Constructive metric-Euclidean ordered planes with free mobility

By a metric-Euclidean ordered plane with free mobility we mean a model of $Cn(\mathcal{A}_2, \text{A3.2.14})$. Its constructive theory will contain the axioms for $\mathcal{CME}$, together with (1.1), A2.1.8, A3.7.7, A3.7.8, with B defined by (3.65). The language in which these axioms are expressed would

[3]The term 'local' is defined as in §3.7.4, with the difference that now $F = Pyth(\mathbb{Q}(t))$ is the Pythagorean hull of $\mathbb{Q}(t)$, i. e. the smallest Pythagorean field containing $\mathbb{Q}(t)$; it can be ordered in only one way such as to extend the given order of $\mathbb{Q}(t)$.

[4]However, we don't know whether the proposed most simple axiom system is completely independent, because we don't know whether the axiom system for $\mathcal{CME}$ (or $\mathcal{CRE}$) is completely independent.

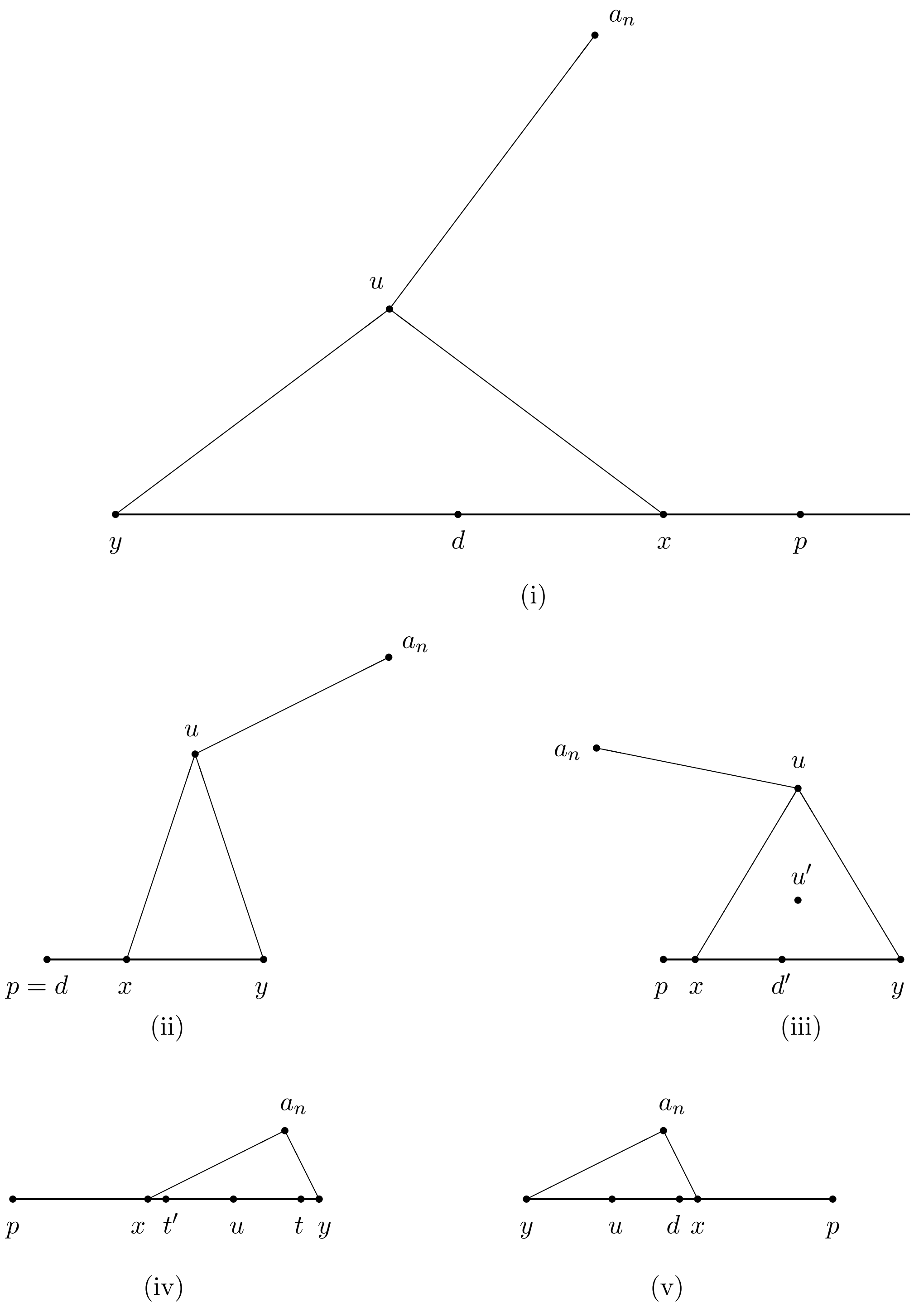

Figure 3.4: The definition of σ from T and U

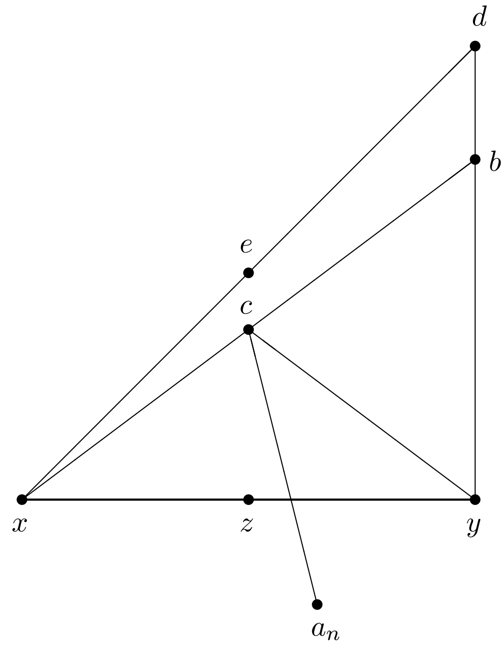

Figure 3.5: The definition of M from T and U

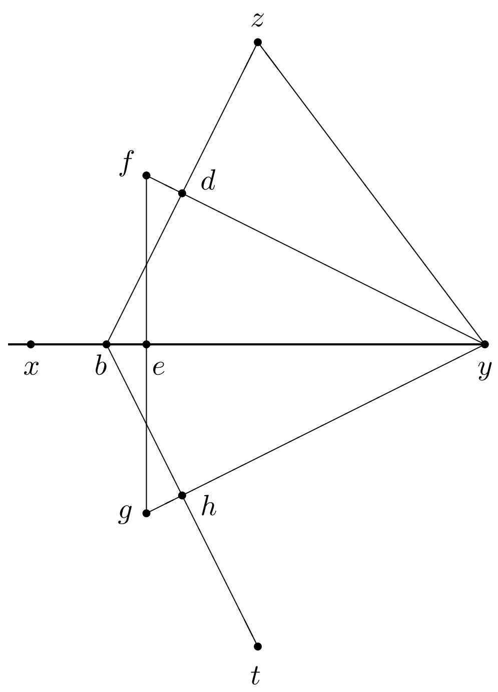

Figure 3.6: The definition of R' from T and U

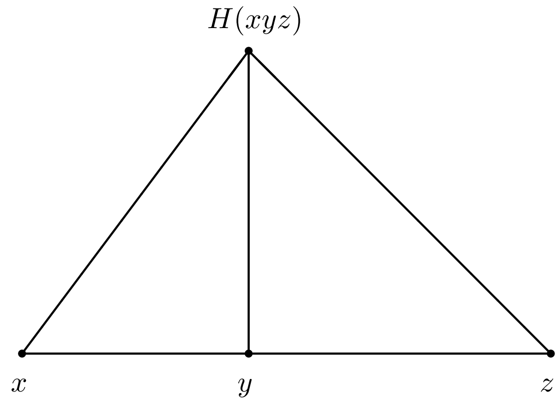

Figure 3.7: The definition of $H(xyz)$

contain R, P, L and T. We shall however choose to consider them expressed in a simpler language, namely in $\mathrm{L}_{fm} = \mathrm{L}(a_0, a_1, a_2, T, M)$. To see that this is possible, notice that
$\sigma(ab) \stackrel{\text{def}}{=} T(abT(M(aM(ab))ba))$
defines σ from T and M, and that R' can now be defined from σ, M and T as in (3.66), so we are able to define R by (3.64), L can be defined as in (3.57), P by (3.63).

Let $\mathcal{FM} \stackrel{\text{def}}{=} Cn_{\mathrm{L}_{fm}}(\mathcal{CME}, (1.1), \mathrm{A}2.1.8, \mathrm{A}3.7.7, \mathrm{A}3.7.8)$. We have (cf. [8, p. 81]):

Representation Theorem 3.7.3 *$\mathfrak{M} \in Mod(\mathcal{FM})$ iff $\mathfrak{M} \simeq \mathfrak{FM}(F)$, where $\mathfrak{FM}(F) = \langle E, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{T}_F, \mathbf{M}_F\rangle$, where $E = \{(a,b) \mid a, b \in M\} \subset \mathbf{A}(F, 0)$, where $M \neq (0)$ is a subgroup of an R-module, that satisfies*

$$\text{for all } u, v, w \in M \ u^2 + v^2 \neq 0 \Rightarrow \frac{u}{\sqrt{u^2+v^2}} w \in M\text{”}.$$

F is a Pythagorean ordered field, $R = \{x \in F \mid (\exists n \in \mathbb{N})\ |x| < n\}$, $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$, $\mathbf{T}_F$ are as in Representation Theorem 3.7.2 and $\mathbf{M}_F(\mathbf{ab}) = (\mathbf{a} + \mathbf{b})/2$.

We ask again the question whether T and M are both needed to axiomatize $\mathcal{FM}$. T is definitely needed, as M would not produce any square roots, but we do not know whether M is needed as well.

We have thus shown that $sd(\mathcal{FM}) = 4$ and hence $Sd([\mathcal{FM}]) = 4$ as well.

3.7.9 The Simplicity degree of $[\mathcal{E}_2']'$ is 4

In order to constructively axiomatize a theory in $[\mathcal{E}_2']'$, we need an additional operation to ensure that a line that passes through a circle intersects it. The operation H will produce the point of intersection of a perpendicular to a diameter of a circle with that circle, provided that the perpendicular intersects the diameter inside the circle (cf. Fig. 3.7), i. e. $H(xyz)$ will have the following intuitive meaning:

$H(xyz) = t$ iff t is the vertex of $\triangle xzt$, right-angled at t, with y the footpoint of the altitude, and such that $(\overrightarrow{yx}, \overrightarrow{yt})$ has the same orientation as $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2})$,
provided that x, y, z are three different points such that $B(xyz)$,
arbitrary, otherwise.

The operation H enables us to express the betweenness relation, if we assume all the axioms for $\mathcal{CD}_2$, i. e. the L_{eu} -axiom system $(\Omega(R,P) \setminus \{(3.19)\}, \mathrm{A3.7.5})$, by stipulating that[5]

$$O(xyz) \leftrightarrow x \neq y \wedge y \neq z \wedge z \neq x \wedge L(xyz) \wedge H(xyz)x \perp H(xyz)z \wedge yH(xyz) \perp yx \tag{3.67}$$

As new axioms we need one stating that $O(\sigma(a_0a_1)a_0x)$ whenever $\mathbf{x}$ is a point on the line $\mathbf{a}_0\mathbf{a}_1$ (which is the line $y = 0$ in $\mathfrak{D}_2(\mathfrak{F}, \mathfrak{k})$ in the standard interpretation) with x-coordinate $a^2 + b^2$, where $a, b \in F$, i. e. an axiom stating that F is a Pythagorean field (this axiom will be A3.7.12). This axiom will also imply that the orthogonality constant k can be normalized to 1, since taking $a = k$ and $b = 0$, it tells us that there is a right triangle with hypotenuse having $(-1, 0)$ and $(k^2, 0)$ as endpoints, and with $(0, 0)$ as footpoint of the altitude. This means that the vertex with the right angle has coordinates $(0, y)$, with $y^2 = k$. In order to state it in a readable way, we need to define the operation $I(xyuv)$ (which was one of the operations used in [46]) and the point with coordinates $(a^2 + b^2, 0)$. The definitions are

$x \neq y \wedge u \neq v \wedge \neg L(uvP(xyu)) \wedge I(xyuv) = z \overset{\text{def}}{\leftrightarrow} x \neq y \wedge u \neq v \wedge \neg L(uvP(xyu))$
$\wedge ((L(xyu) \wedge u = z) \vee (L(xyv) \wedge v = z) \vee (L(uvx) \wedge x = z) \vee (L(uvy) \wedge y = z)$
$\vee (\neg L(uvx) \wedge \neg L(xyR'(uvx)) \wedge U(xR'(uvx)R'(xyR'(uvx))) = z) \vee (\neg L(uvx)$
$\wedge \neg L(xyR'(uv\sigma(ux))) \wedge U(\sigma(ux)R'(uv\sigma(ux))R'(xyR'(uv\sigma(ux)))) = z))$,
and, with $i = i(x) \overset{\text{def}}{=} I(a_0a_2xP(a_1a_2x)))$,
$L(a_0a_1x) \wedge s(x) = z \overset{\text{def}}{\leftrightarrow} L(a_0a_1x) \wedge I(a_0a_1iP(a_2xi)) = z$,
$L(a_0a_1x) \wedge L(a_0a_1y) \wedge Q(x, y) = z \overset{\text{def}}{\leftrightarrow} L(a_0a_1x) \wedge L(a_0a_1y) \wedge P(a_0s(x)s(y)) = z$.

The first definition 'says' that in order to find the point of intersection of the lines xy and uv (if it is not already one of x, y, u, v), one needs only to find a point that does not lie on uv and whose reflection in uv does not lie on xy (which is why we distinguish two cases depending upon whether $xy \perp uv$ (and then $\sigma(ux)$ is the right point) or $\neg\, xy \perp uv$ (in which case x is the right point)); for then that point, say p, its reflection in uv, say p', the reflection of p' in xy, say p'', form a (non-degenerate) triangle $\triangle pp'p''$, whose circumcentre $U(pp'p'')$ is the intersection of xy and uv.

The second one 'says' that, if $\mathbf{x}$ is a point on the line $\mathbf{a}_0\mathbf{a}_1$, which in the standard interpretation is the line $y = 0$, having coordinates $(a, 0)$, then $\mathbf{z}$ is the point whose coordinates are $(a^2, 0)$.

The third one 'says' that, if $\mathbf{x}$ and $\mathbf{y}$ are two points on the line $\mathbf{a}_0\mathbf{a}_1$, having coordinates $(a, 0)$ and $(b, 0)$ respectively, then $\mathbf{z}$ is the point whose coordinates are $(a^2 + b^2, 0)$.

We also need an axiom (A3.7.13) stating that among three different points on a line one is between the other two, as well as axioms (A3.7.14-A3.7.16) that choose one of the two intersections of the perpendicular in y to the line xz with the circle having xz as diameter

[5] $O(xyz)$ may be read as 'y lies strictly between x and z'.

(for x, y, z different and such that y lies between x and z). The choice is made such that $(\overrightarrow{yx}, \overrightarrow{yt})$ has the same orientation as $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2})$.

With $a_0' = F(a_0a_1a_2)$, $a_2' = H(a_1a_0'\sigma(a_0'a_1))$ and $Hl(abc) \leftrightarrow O(abc) \vee O(acb) \vee b = c$, we are now ready to state the new axioms:

A 3.7.12 $L(a_0a_1x) \wedge L(a_0a_1y) \to O(\sigma(a_0a_1)a_0Q(x,y))$,

A 3.7.13 $L(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a \to O(abc) \vee O(bca) \vee O(cab)$,

A 3.7.14 $Hl(a_0'a_2H(a_1a_0'\sigma(a_0'a_1)))$,

A 3.7.15 $a_1P(yxa_0') \equiv a_2'P(yH(xy\sigma(yx)a_o') \wedge \neg(Hl(a_0'a_2'P(yxa_0'))$
$\wedge Hl(a_0'a_1H(xy\sigma(yx))) \wedge \neg(O(a_2'a_0'P(yxa_0')) \wedge O(a_1a_0'H(xy\sigma(yx))))$

A 3.7.16 $O(abc) \to Hl(bH(ab\sigma(ba))H(abc))$.

Note that $B(abc)$ iff $O(abc) \vee a = b \vee b = c$, and that in order to show that the coordinate field is ordered by O, all we need to do is to show that A2.1.7, A2.1.8, (1.1), A2.1.9 can be deduced from $\mathcal{CD}_2$, (3.67), A3.7.12-A3.7.16, with B defined from O, which in turn is defined by (3.67). To see this, notice that the coordinate fields of the models of $Cn(\mathcal{CD}_2, \text{A3.7.12})$ are formally real, since they are 2-formally real (because $k = 1$, this follows from the representation theorem for $\mathcal{CD}_2'$) and Pythagorean. Now all of the order axioms we have to prove follow from A3.7.13 and the fact that the coordinate fields are formally real. Let's prove, for example, $O(abc) \to O(cba)$. Suppose $O(abc)$. By (3.67) we have $L(abc)$, hence $L(cba)$ (by A3.2.2). Since a, b, c are different, we can apply A3.7.13 to deduce that $O(cba) \vee O(bac) \vee O(acb)$. A simple computation shows that $O(abc) \wedge O(bac)$, as well as $O(abc) \wedge O(acb)$ lead to a contradiction, as they imply that a sum of squares of non-zero elements is zero. Therefore we must have $O(cba)$.

Let $\mathrm{L}_{e'} = \mathrm{L}(a_0, a_1, a_2, U, R', H)$ and let $\mathcal{CE}_2' \stackrel{\text{def}}{=} Cn_{\mathrm{L}_{e'}}(\mathcal{CD}_2, \text{A3.7.12-A3.7.16})$. We have shown that

Representation Theorem 3.7.4 *$\mathfrak{M} \in Mod(\mathcal{CE}_2')$ iff $\mathfrak{M} \simeq \mathfrak{K}_2(F)$, where F is an ordered Euclidean field and $\mathfrak{K}_2(F) = \langle F \times F, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{R'}_F, \mathbf{U}_F, \mathbf{H}_F\rangle$, with $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$, $\mathbf{R}_F' = \mathbf{R}_{(F,1)}'$, $\mathbf{U}_F = \mathbf{U}_{(F,1)}$ defined as in Representation Theorem 3.7.1, and*

$$\mathbf{H}_F(\mathbf{abc}) = \begin{cases} (t_1, t_2), \textit{ which is the solution of the system} \\ (z_2 - t_2)(x_2 - t_2) - (t_1 - z_1)(x_1 - t_1) = 0, \\ (t_2 - y_2)(x_2 - y_2) - (t_1 - y_1)(y_1 - x_1) = 0, \\ (x_1 - y_1)(t_2 - y_2) - (x_2 - y_2)(t_1 - y_1) > 0, \\ \textit{if } \mathbf{x} \neq \mathbf{y}, \mathbf{y} \neq \mathbf{z}, (x_1 - y_1)(x_2 - z_2) - (x_2 - y_2)(x_1 - z_1) = 0, \\ (x_1 - y_1)(y_1 - z_1) \geq 0 \textit{ and } (x_2 - y_2)(y_2 - z_2) \geq 0, \\ \textit{arbitrary, otherwise.} \end{cases}$$

About the question whether $\mathcal{CE}_2'$ can be constructively axiomatized in a strict sublanguage of $\mathrm{L}_{e'}$, we mention that both U and H are definitely needed: U is needed because both R' and

H are local operations[6], and H is needed because U and R' would not produce any square root (so if the $\mathbf{a}_i$'s are all rational, then their closure under U and R' is included in $\mathbb{Q} \times \mathbb{Q}$). We do not know whether R' is needed as well, but conjecture that we do indeed need it.

A different axiom system for a constructive theory synonymous to $\mathcal{CE}'_2$, some of whose axioms require more than 4 variables, axiomatized in $\mathrm{L}(a_0, a_1, a_2, U, T, H)$ was given in [51], [52]. Since R' can be defined from U and T, as in (3.66), the simple axiom system proposed here can also be expressed in $\mathrm{L}(a_0, a_1, a_2, U, T, H)$. We considered it as an axiom system in $\mathrm{L}_{e'}$, because we find that the purely metric operation R' is preferable to the operation T that involves, beside a metric concept, both order and free mobility.

3.8 Mehr Licht

All the results obtained so far may seem to be saying little about geometry as such, being mostly concerned with how one talks about geometry.

We now want to show that this analysis sheds new light upon the very essence of Euclidean geometry. Let $\mathrm{L}_t = \mathrm{L}(a_0, a_1, a_2, R', R, F, P, U, O_B, T, H)$. We now know that R, F, O_B, P, T can be defined from U, R' and H; we denote by *Def* the set of these definitions[7], and let $\overline{\mathcal{CE}'_2} \stackrel{\text{def}}{=} Cn_{\mathrm{L}_t}(\mathcal{CE}'_2, Def)$. That is, $\overline{\mathcal{CE}'_2}$ is constructive plane Euclidean geometry in which all ruler and compass constructions can be performed, in which we have names for the operations R', R, F, P, U, O, T, H and for three distinguished points a_0, a_1, a_2. Let $\mathrm{L}_{mt} = \mathrm{L}(a_0, a_1, a_2, R', R, F, P, U)$, let *def* be the set of the definitions of R, F, P, from U and R', and let $\overline{\mathcal{CD}_2} \stackrel{\text{def}}{=} Cn_{\mathrm{L}_{mt}}(\mathcal{CD}_2, def)$. For $n \in \mathbb{N}$, let τ_n be the universal L_{eu}-sentences stating that $x_1^2 + \cdots + x_n^2 \neq -1$. According to the representation theorems proved so far, we have[8]

Theorem 3.8.1
(i) $\mathcal{CE}_2 = Cn((\overline{\mathcal{CE}'_2} \cap \mathrm{L}_e)_\forall)$,
(ii) $\mathcal{CBD}_2 = Cn((\overline{\mathcal{CE}'_2} \cap \mathrm{L}_{oe})_\forall)$,
(iii) $\mathcal{FM} = Cn((\overline{\mathcal{CE}'_2} \cap \mathrm{L}_{fm})_\forall)$,
(iv) $Cn(\mathcal{CD}_2 \cup \{\tau_n | n \in \mathbb{N}\}) = Cn((\overline{\mathcal{CE}'_2} \cap \mathrm{L}_{eu})_\forall)$,
(v) $\mathcal{CME} = Cn((\overline{\mathcal{CD}_2} \cap \mathrm{L}_{me})_\forall)$,
(vi) $\mathcal{CRE} = Cn((\overline{\mathcal{CD}_2} \cap \mathrm{L}_{re})_\forall)$.

This theorem says that, for example, constructive plane metric-Euclidean geometry is what we get if we are in the geometry of Euclidean planes, but allowed to speak only in terms of F and P, never being allowed to utter 'there is'.

Proof. We shall prove only (iv), the other cases being similar and easier to prove. The inclusion $Cn(\mathcal{CD}_2 \cup \{\tau_n | n \in \mathbb{N}\}) \subseteq Cn((\overline{\mathcal{CE}'_2} \cap \mathrm{L}_{eu})_\forall)$ is easily checked. To prove the reverse

[6]The term 'local' is defined as in §3.7.4, with the difference that now $F = Eu(\mathbb{Q}(t))$ is the Euclidean hull of $\mathbb{Q}(t)$, i. e. the smallest Euclidean field containing $\mathbb{Q}(t)$; it can be ordered in only one way such as to extend the given order of $\mathbb{Q}(t)$.

[7]Note that we define the operations O_B and T only for those arguments for which the operations have a geometric meaning, i. e. are not 'arbitrary'.

[8]$\mathcal{T}_\forall$ stands for the theory containing all the universal sentences in $\mathcal{T}$.

inclusion, suppose that there is a universal sentence σ in $Cn((\overline{\mathcal{CE}'_2} \cap \mathrm{L}_{eu})_\forall)$ but not in $Cn(\mathcal{CD}_2 \cup \{\tau_n | n \in \mathbb{N}\})$. Since $\neg\sigma$ and $Cn(\mathcal{CD}_2 \cup \{\tau_n | n \in \mathbb{N}\})$ are consistent, there is a model $\mathfrak{M}$ satisfying both. Being a model of $Cn(\mathcal{CD}_2 \cup \{\tau_n | n \in \mathbb{N}\})$, $\mathfrak{M}$ is isomorphic to a constructive Euclidean plane $\mathfrak{E}_2(F, k)$ over some formally real field F. Let $\overline{F}$ denote the real closure of F, considered as an ordered field with the unique order definable in it. Now $(Th_{\mathrm{L}_{e'}}(\mathfrak{K}_2(\overline{F})))_\forall$ must be an extension of $(\overline{\mathcal{CE}'_2} \cap \mathrm{L}_{eu})_\forall$, but it also satisfies $\neg\sigma$, a contradiction. $\square$

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Chapter 4

THE ALGORITHMIC SETTING

So fern hab ich mir nie die Ewigkeit gedacht...
Es weinen über unsere Welt die Engel in der Nacht.
Else Lasker-Schüler, *Aus der Ferne.*

4.1 Introduction

So far all our discussion has been going on in first-order logic, with one short escapade into the infinitary logic $L_{\omega_1\omega}$. The constructive setting of chapter 3 suggests a meaningful logic in which to address constructibility problems that may require a finite, but not a priori bounded, number of constructions. We are, for example, able to determine constructively that two given segments 'behave Archimedeanly' (i. e. that an integer multiple of the length of either of them exceeds the length of the other), by laying off, in increasing order, integer multiples of one on the other from one of the latter's endpoints. If we get past the endpoint of the 'longer' segment, we stop, if not, we continue. If the logic allows us to state that such constructions terminate after finitely many steps, then we are able to express the Archimedeanity of the coordinate field. At first, this may seem that what we are asking for is $L_{\omega_1\omega}$ or, maybe its constructive fragment $CL_{\omega_1\omega}$ (cf. [41]). It turns out, however, that one may restrict one's attention to a much more restrictive part than even the universal fragment of $CL_{\omega_1\omega}$, namely to a quantifier-free logic, containing only Boolean combinations of halting-formulas for flow-charts (that may contain loops but not recursive calls). Such a logic was introduced by E. Engeler [18] under the name of *algorithmic logic* and its relevance to geometry was studied in [19], [20], [21].

We shall investigate in this chapter axiomatizations in algorithmic logic of the various constructive geometries discussed so far. The language will still contain only at most ternary operation symbols and the axioms will still be only universal statements; the difference will be that they will be allowed to state that a flow-chart program terminates.

4.2 The algorithmic logic

We shall repeat the definition of algorithmic logic as given in [73] in our context of a language without relation symbols.

We begin with a formal description of flow-charts, which may be thought of as trees consisting, besides exactly one root and at least one leaf, of two kinds of nodes: nodes at which there is one input and one output, and nodes at which there is one input and two outputs. At the first kind of nodes a variable is assigned a certain value, at the second kind of nodes the question whether the input is equal to a certain value is asked and the outcome (yes, no) determines which path is to be followed.

A *directed graph* is a relational structure $\mathbf{G} = \langle V_{\mathbf{G}}, E_{\mathbf{G}} \rangle$, where $E_{\mathbf{G}} \subseteq V_{\mathbf{G}} \times V_{\mathbf{G}}$ (the elements of $V_{\mathbf{G}}$ will be called *vertices* and $e = (v, v')$ with $e \in E_{\mathbf{G}}$ (which will be denoted by $E_{\mathbf{G}}(v, v')$) will be called an *edge*). For $v \in V_{\mathbf{G}}$ let $dg^+(v) = |\{v' \in V_{\mathbf{G}} | E_{\mathbf{G}}(v, v')\}|$ and $dg^-(v) = |\{v' \in V_{\mathbf{G}} | E_{\mathbf{G}}(v', v)\}|$. A sequence of vertices $p = (v_1, \ldots, v_n)$ with $v_i \in V_{\mathbf{G}}$ $(i = 1, \ldots, n)$ will be called a *path* in $\mathbf{G}$ if $E_{\mathbf{G}}(v_i, v_{i+1})$ for $i = 1, \ldots, n-1$. The vertices $v \in V_{\mathbf{G}}$ with $dg^-(v) = 0$ (respectively $dg^+(v) = 0$) will be denoted by 'S' for 'start' (respectively 'H' for 'halt').

A finite directed graph (i.e $V_{\mathbf{G}}$ is a finite set) will be called a *flow-chart* if

(i) there is exactly one $v \in V_{\mathbf{G}}$ with $dg^-(v) = 0$;

(ii) there is at least one $v \in V_{\mathbf{G}}$ with $dg^+(v) = 0$;

(iii) for all $v \in V_{\mathbf{G}}$ with $dg^+(v) \neq 0$ either $dg^+(v) = 1$ or $dg^+(v) = 2$.

Let $\mathrm{L} = \mathrm{L}(\mathbf{F}, r)$ be a first-order language, where $\mathbf{F}$ is a finite set of operation symbols, and $r : \mathbf{F} \to \mathbb{N}$ is a function assigning to each $f \in \mathbf{F}$ its arity $r(f)$. Let $\mathbf{V}$ be the set of variables for L. Let

$\Sigma_{\mathrm{L}}^{\mathbf{F}}$ be the set of assignments of the form $x \leftarrow \tau$ where $\tau = x_1$, or $\tau = f(x_1 \ldots x_{r(f)})$ with $x_1, \ldots, x_{r(f)}, x \in \mathbf{V}$, and $f \in \mathbf{F}$;

Σ_{L}^{f} be the set of quantifier-free formulas $\varphi(x_1 \ldots x_n)$ with free variables $x_1 \ldots x_n$ $(n \in \mathbf{N})$ and $x_1 \ldots x_n \in \mathbf{V}$; [1]

Σ_{L} be $\Sigma_{\mathrm{L}}^{\mathbf{F}} \cup \Sigma_{\mathrm{L}}^{f}$.

For any sequence of symbols α (formula or assignment) let $V(\alpha)$ be the set of all variables in α.

Let $\mathbf{G}$ be a flow-chart, $j : E_{\mathbf{G}} \to \Sigma_{\mathrm{L}}$ a map and $V_n = \{x_1, \ldots x_n\}$ with $x_1, \ldots x_n \in \mathbf{V}$. $\Pi_{\mathrm{L}}(x_1, \ldots x_n) = (\mathbf{G}_\Pi, j_\Pi, V_\Pi) = (\mathbf{G}, j, V_n)$ will be called a *program* over Σ_{L} in the variables $x_1, \ldots x_n$ if

(i) for $v \in V_{\mathbf{G}}$ with $dg^+(v) = 1$ there is an $v' \in V_{\mathbf{G}}$ and $\alpha \in \Sigma_{\mathrm{L}}^{\mathbf{F}}$ with $E_{\mathbf{G}}(v, v'), j(v, v') = \alpha$ and $V(\alpha) \subseteq V_\Pi$;

(ii) for $v \in V_{\mathbf{G}}$ with $dg^+(v) = 2$ there are $v', v'' \in V_{\mathbf{G}}$ and $\alpha \in \Sigma_{\mathrm{L}}^{f}$ with $E_{\mathbf{G}}(v, v')$, $E_{\mathbf{G}}(v, v''), j(v, v') = \alpha, j(v, v'') = \neg\alpha$ and $V(\alpha) \subseteq V_\Pi$.

The map j_Π thus establishes a correspondence between the paths p in $\mathbf{G}_\Pi$ and sequences $w_\Pi(p)$ of elements from $\Sigma_\Pi \overset{\text{def}}{=} \{j(v, v') \in \Sigma_{\mathrm{L}} | (v, v') \in E_{\mathbf{G}_\Pi}\}$, which we interpret as words over

[1] We could have taken Σ_{L}^{f} to be the set of formulas $x_i = x_j$ and $\neg(x_i = x_j)$ with $x_i, x_j \in \mathbf{V}$. The drawback would have been that programs would have become longer.

Σ_Π. Let $W(\Pi) \stackrel{\text{def}}{=} \{w_\Pi(p)|p$ a path in $\mathbf{G}_\Pi\}$ and $W_{SH}(\Pi) \stackrel{\text{def}}{=} \{w_\Pi(p)|p$ is a path from S to H in $\mathbf{G}_\Pi\}$.

For a given program $\Pi_{\mathrm{L}}(x_1,\ldots,x_n)$, we define a quantifier-free formula ϕ_w (describing under what conditions on x the program $\Pi_{\mathrm{L}}(x_1,\ldots,x_n)$ will follow the path w) with $V(\phi_w) \subseteq V_\Pi$ inductively over $w \in W(\Pi)$.

(i) If $\lambda \in W(\Pi)$ is the empty word, then $\phi_\lambda(x_1,\ldots,x_n) = (x_1 = x_1 \wedge \ldots \wedge x_n = \ x_n)$;

(ii) if $\alpha \cdot w \in W(\Pi)$ with $\alpha \in \Sigma^{\mathbf{F}}_{\mathrm{L}}, \alpha = x \leftarrow \tau$ and $V(\alpha) \subseteq V(\Pi)$, then $\phi_{\alpha\cdot w}(x_1,\ldots,x_n)$ is the formula obtained by substituting τ for x in $\phi_w(x_1,\ldots,x_n)$;

(iii) if $\alpha \cdot w \in W(\Pi)$ with $\alpha \in \Sigma^f_{\mathrm{L}}$, and $V(\alpha) \subseteq V(\Pi)$, then $\phi_{\alpha\cdot w}(x_1,\ldots,x_n) = \alpha \wedge \phi_w(x_1,\ldots,x_n)$.

ϕ_Π will be called a *halting formula* for a program $\Pi_{\mathrm{L}}(x_1,\ldots,x_n)$ if

$$\phi_\Pi(x_1,\ldots,x_n) = \bigvee_{w\in W_{SH}(\Pi)} \phi_w(x_1,\ldots,x_n).$$

ϕ_Π will in general no longer be a first-order formula, but one in $\mathrm{L}_{\omega_1\omega}$.

The *algorithmic language* $aL(\mathrm{L})$ is the least language over L for which

(i) for every program Π over Σ_{L}, $\phi_\Pi \in aL(\mathrm{L})$;

(ii) if $\alpha \in aL(\mathrm{L})$, then $\neg\alpha \in aL(\mathrm{L})$;

(iii) if $\alpha, \beta \in aL(\mathrm{L})$, then $\alpha \vee \beta \in aL(\mathrm{L})$.

$aL(\mathrm{L})$ is a sublanguage of $\mathrm{L}_{\omega_1\omega}$[2] (but not of L). Let $\mathfrak{M}$ be a structure for L and let Σ denote a set of sentences in $aL(\mathrm{L})$. We denote by $aTh(\mathfrak{M}) \stackrel{\text{def}}{=} \{\alpha \in aL(\mathrm{L})|\, \mathfrak{M} \models \alpha\}$ the *algorithmic theory of* $\mathfrak{M}$ and by $aCn(\Sigma) \stackrel{\text{def}}{=} \{\alpha \in aL(\mathrm{L})|\, \Sigma \models \alpha\}$ the set of *algorithmic consequences* of Σ.

4.3 Ruler and compass constructions

We have seen in §3.7.9 that one can axiomatize constructively a theory in $[\mathcal{E}'_2]$ in the language $\mathrm{L}(a_0, a_1, a_2, U, T, H)$ and we know that we cannot axiomatize by quantifier-free axioms constructive Euclidean planes over Euclidean ordered fields using only T and H, since these operations are 'local' and we cannot reach points that lie 'far way' using only local operations.

However, it was observed as early as 1672, by G. Mohr [45], that in Archimedean ordered Cartesian planes over Euclidean fields one can perform all ruler and compass constructions using the compass alone. The compass being a 'local' instrument, this tells us the situation changes drastically in the Archimedean case, and that it is precisely the existence of 'far away' points (i. e. of infinitely large elements of the coordinate field) that requires a specifically Euclidean operation, like Moler-Suppes's I or our U, in the language of a constructive axiom system.

The Mohr-Mascheroni theorem (cf. [39] or [67] for a proof; a more general version has been proved by A. Avron in [3], [4]; the name of the theorem is justified by [45] and [44]) may be rephrased, by strengthening its statement to allow only the use of 'collapsing compasses', i. e. we are allowed to use the compass only for drawing circles, not for the

[2] In fact a sublanguage of $\mathrm{CL}_{\omega_1\omega}$.

laying off of segments, in our algorithmic context to assert that plane Euclidean geometry over Archimedean ordered Euclidean fields can be axiomatized by axioms in $aL(\mathrm{L}_1)$, where $\mathrm{L}_1 = \mathrm{L}_1(a_0, a_1, a_2, IC)$, with a_0, a_1, a_2 to be interpreted as three non-collinear points, and $IC(xyuv) = z$ iff 'z is the intersection point of the circles with centres x and u and radii xy and uv respectively (that intersection point for which $(\overrightarrow{uz}, \overrightarrow{ux})$ has the same orientation as $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2})$ if the circles intersect in two points), provided that $u \neq v \wedge x \neq y$ and the two circles intersect, and $z = x$, otherwise'.

We are thus naturally led to the question whether we can axiomatize Euclidean geometry over Archimedean ordered Euclidean fields in some algorithmic logic of a language that contains only 'local', 'absolute' ternary operations.

The language that provides an affirmative answer to this question is L= $\mathrm{L}(a_0, a_1, a_2, T', H)$, where the intended interpretation of T' is slightly different from that of T, namely $T'(abc) = d$ if 'd is as distant from a on the ray $\overrightarrow{ca}$ as b is from a, provided that $a \neq c \vee (a = c \wedge a = b)$, and arbitrary, otherwise'.

For any Euclidean ordered field F, let $\mathfrak{E}(F) = \langle F \times F, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{T}'_F, \mathbf{H}_F \rangle$, with $\|\mathbf{x}\| = x_1^2 + x_2^2$, $\mathbf{H}_F$ as in Representation Theorem 3.7.4,
$\mathbf{a}_0 = (0,0)$, $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (\alpha, \beta)$, and

$$\mathbf{T}'_F(\mathbf{xyz}) = \begin{cases} (x_1 + AB^{-1}(x_1 - z_1), x_2 + AB^{-1}(x_2 - z_2)), \\ \text{ with } A = \sqrt{\|\mathbf{x} - \mathbf{y}\|}, B = \sqrt{\|\mathbf{x} - \mathbf{z}\|} \text{ if } B \neq 0, \\ \mathbf{x} \text{ if } A = B = 0, \\ \text{ arbitrary, otherwise }, \end{cases}$$

We shall give an axiom system for $aTh(\mathfrak{E}(\mathbb{R}))$ and prove that every model of that system will be a model of plane geometry over some Archimedean ordered Euclidean field.

4.3.1 Construction of the midpoint

We shall show how to construct the midpoint of a segment ab using only the operations T' and H. The construction can be followed by looking at Fig. 4.1 (if the segment ab is of unit length, then the length of $b\psi_6$ is $\sqrt{61}$, the length of $b\iota$ and $b\eta$ is $\sqrt{63}$, and the length of $a\iota$ is 8)

$\sigma = \sigma(ab) \stackrel{\text{def}}{=} T'(abb)$, $\quad T(abc) \stackrel{\text{def}}{=} T'(ab\sigma(ac))$,
$\varphi_0 = \varphi_0(ab) \stackrel{\text{def}}{=} b$; $\varphi_1 = \varphi_1(ab) \stackrel{\text{def}}{=} a$; $\varphi_{n+2} = \varphi_{n+2}(ab) \stackrel{\text{def}}{=} \sigma(\varphi_{n+1}\varphi_n)$ for $n = 0, 1, \ldots$,
$\alpha = \alpha(ab) \stackrel{\text{def}}{=} H(\varphi_6\varphi_5\varphi_4)$,
$\psi_0 = \psi_0(ab) \stackrel{\text{def}}{=} \varphi_5$; $\psi_1 = \psi_1(ab) \stackrel{\text{def}}{=} \alpha$; $\psi_{n+2} = \psi_{n+2}(ab) \stackrel{\text{def}}{=} \sigma(\psi_{n+1}\psi_n)$ for $n = 0, 1, \ldots$,
$\beta = \beta(ab) \stackrel{\text{def}}{=} T'(\psi_6\psi_5 b)$, $\quad \gamma = \gamma(ab) \stackrel{\text{def}}{=} \sigma(\psi_6\beta)$,
$\delta = \delta(ab) \stackrel{\text{def}}{=} H(\gamma\psi_6\beta)$, $\quad \epsilon = \epsilon(ab) \stackrel{\text{def}}{=} \sigma(\gamma\psi_6)$,
$\zeta = \zeta(ab) \stackrel{\text{def}}{=} H(\psi_6\gamma\epsilon)$, $\quad \eta = \eta(ab) \stackrel{\text{def}}{=} T'(\psi_6\zeta\delta)$,
$\theta = \theta(ab) \stackrel{\text{def}}{=} H(\sigma(ba)ba)$, $\quad \iota = \iota(ab) \stackrel{\text{def}}{=} T'(b\eta\theta)$,
$\kappa = \kappa(ab) \stackrel{\text{def}}{=} T(a\varphi_3\iota)$, $\quad \lambda = \lambda(ab) \stackrel{\text{def}}{=} \sigma(\kappa a)$,
$\mu = \mu(ab) \stackrel{\text{def}}{=} T(b\varphi_2\lambda)$, $\quad \nu = \nu(ab) \stackrel{\text{def}}{=} T(b\varphi_3\lambda)$,
$\kappa' = \kappa'(ab) \stackrel{\text{def}}{=} \sigma(\nu\kappa)$,

$$M(ab) \stackrel{\text{def}}{=} \begin{cases} \sigma(\mu\kappa') & \text{for } a \neq b \\ a & \text{for } a = b. \end{cases}$$

4.3.2 Construction of the reflection of a point in a line

As in [51], let
$P(abc) \stackrel{\text{def}}{=} \sigma(M(bc)a), \qquad S(abcd) \stackrel{\text{def}}{=} T(cP(abc)d),$
$S(abcd)$ being the result of laying off ab from c on ray $\overrightarrow{cd}$, and define the following relations: B by (3.65), L by (1.1) and $\equiv$ by
$ab \equiv cd \stackrel{\text{def}}{\leftrightarrow} (c \neq d \wedge S(abcd) = d) \vee (a = b \wedge c = d).$
Let $F_0(bca)$ stand for the footpoint of the altitude from a in $\triangle abc$ (provided that $\neg L(abc)$; arbitrary, otherwise), i.e. let (cf. Fig. 4.2):
$m = m(abc) \stackrel{\text{def}}{=} M(aT(bac)),$
$n = n(abc) = T(bmc),$
$p_1 = p_1(abc) \stackrel{\text{def}}{=} H(bnT(bac)); \ p_2 = p_2(abc) \stackrel{\text{def}}{=} H(T(bac)nb),$
$q_i = q_i(abc) \stackrel{\text{def}}{=} S(T(bac)mnp_i)$, for $i = 1, 2$,
$r_i = r_i(abc) \stackrel{\text{def}}{=} T(bmq_i)$, for $i = 1, 2$,
$\neg L(abc) \wedge f(abc) = d \stackrel{\text{def}}{\leftrightarrow} \bigvee_{i=1}^{2}(d = M(mr_i) \wedge \neg(r_i = m)),$
$g = g(abc) \stackrel{\text{def}}{=} P(maf(abc)),$
$F = F_0(bca) \stackrel{\text{def}}{=} \sigma(ga).$
Now, for $b \neq c$

$$R'(bca) \stackrel{\text{def}}{=} \begin{cases} \sigma(Fa) & \text{if } \neg L(abc) \\ a & \text{if } L(abc) \end{cases}$$

gives the reflection of the point a in the line bc.

4.3.3 Construction of the centre of the circumcircle

We begin by describing informally how to construct the circumcentre. Formal definitions that show how the constructions can be performed by using only T' and H will follow.

The construction of the centre of the circumcircle of a triangle abc (where we denote the lengths of its sides by $|cb| = x$,$|ca| = y$, $|ab| = z$) can be descibed by the following program:

First see if the triangle is acute, right, or obtuse (this can be checked, since a triangle is acute, right, respectively obtuse in the vertex c if and only if $y^2 + x^2$ is $>, =,$ resp. $< z^2$).

If the triangle is right, the centre of the circumcircle is the midpoint of the hypotenuse.

Suppose the triangle is obtuse. Then transport (using the operation 'S') a segment of length $\sqrt{R^2 - \frac{z^2}{4}}$ with one of its endpoints the midpoint of ab on the perpendicular bisector of the side opposite to the obtuse angle (say $\hat{c}$), in the halfplane that does not contain c. Here R stands for the radius of the circumcircle, and it is given by $R = \frac{xyz}{4A}$, where A is the area of $\triangle abc$, which, according to Heron's formula is $\sqrt{p(p-x)(p-y)(p-z)}$, where $p = \frac{x+y+z}{2}$. Since, with the aid of the operation 'H' we can construct $\sqrt{uv}$ whenever u and v are the

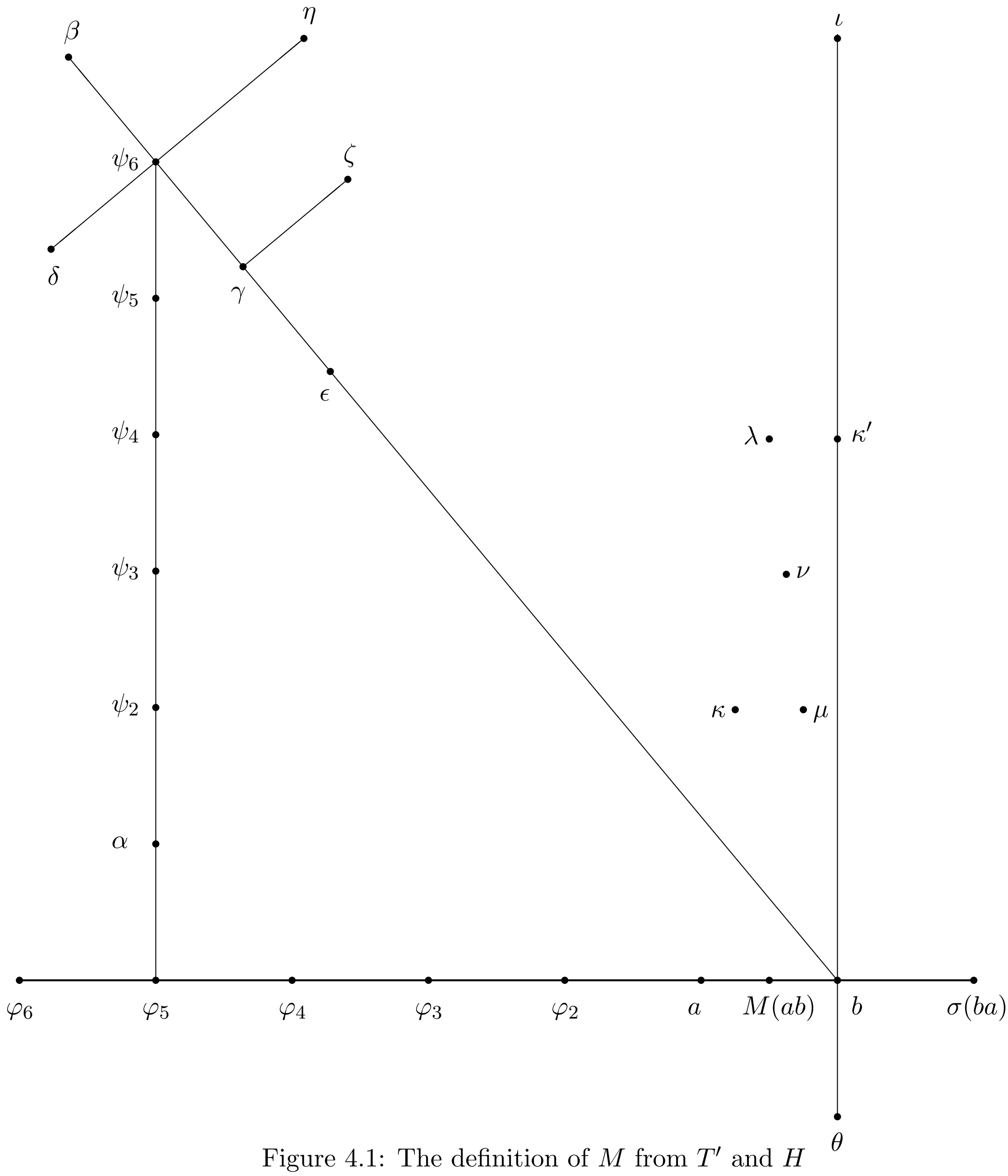

Figure 4.1: The definition of M from T' and H

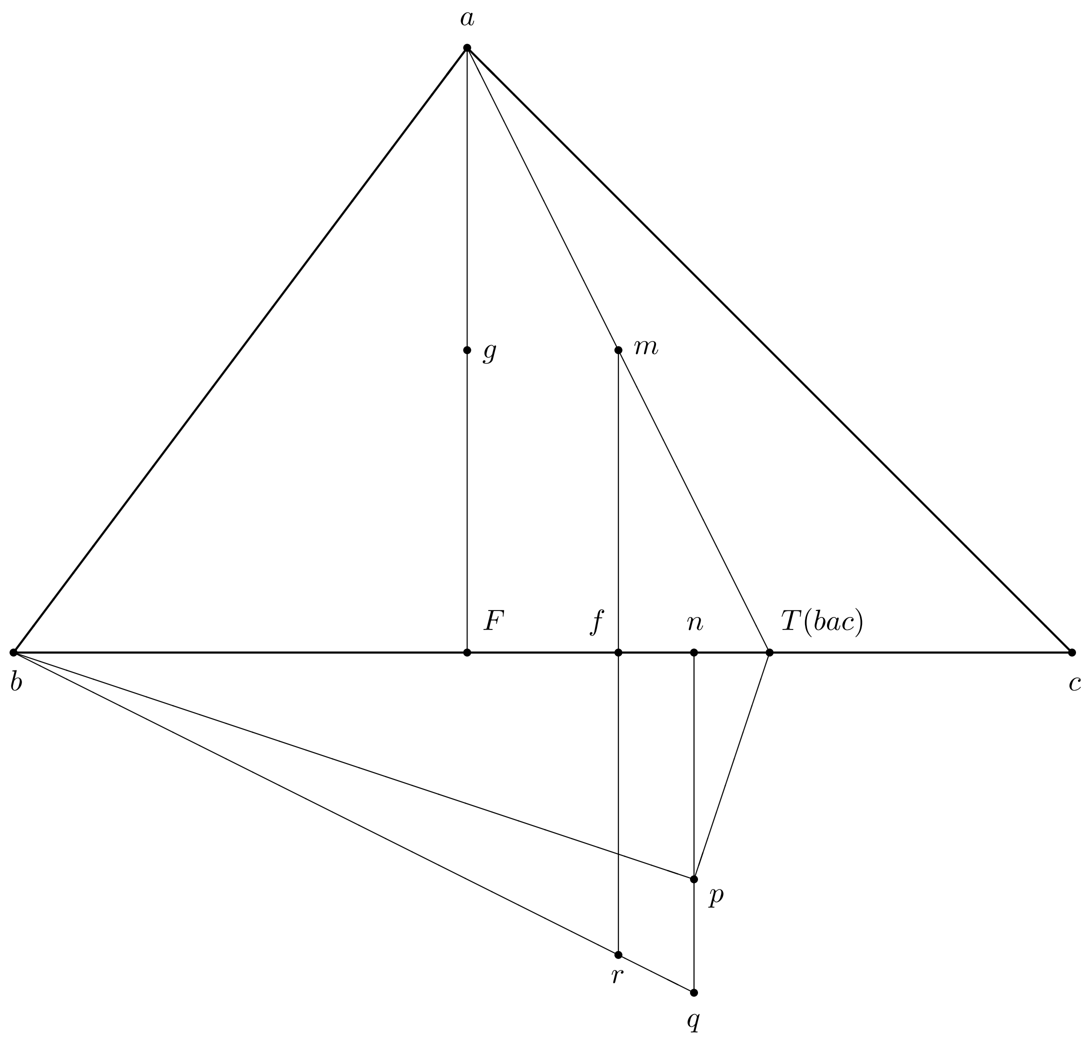

Figure 4.2: The definition of R' from T' and H

lengths of given segments, A is in fact the product of two segment-lengths, say x' and y'. So we have to construct $R = \frac{xyz}{4x'y'}$. In order to construct a segment of length R, it is enough to show how to construct a segment $\frac{uv}{w}$, where u, v, w are the lengths of given segments.

First construct $w' = \sqrt{uv}$.

Check if $w > w'$.

If it is, then construct $\sqrt{w^2 - w'^2} = \sqrt{(w - w')(w + w')}$, and then the right triangle with sides of lengths $w', \sqrt{w^2 - w'^2}$, and w. Construct the height in this right triangle. The projection of side of length w' on the hypotenuse will have length $\frac{w'^2}{w} = \frac{uv}{w}$, and we are done.

If $w < w'$, then, according to the Archimedean axiom, there is an $n \in \mathbb{N}$ such that $nw > w'$. Choose such an n, and then construct as above $\frac{uv}{nw}$. But then $n \cdot \frac{uv}{nw} = \frac{uv}{w}$ can also be contructed, and we are done.

Now that we have shown how to construct a segment of length R, we can also construct a segment of length $\sqrt{R^2 - \frac{z^2}{4}} = \sqrt{(R - \frac{z}{2})(R + \frac{z}{2})}$, and hence $U(abc)$.

If the triangle is acute, do the same constructions as in the obtuse case with the only difference that the transport of the segment of lenth $\sqrt{R^2 - \frac{z^2}{4}}$ will be done in the halfplane containing c.

We now proceed to formally define the operation U for three non-collinear points in terms of T' and H.

Let $j(abc)$ stand for 'abc is a triangle with sides ab and ac not greater than bc', i.e.
$j(abc) \overset{\text{def}}{\leftrightarrow} \neg L(abc) \wedge ((B(aT(abc)c) \wedge B(cT(cab)b)) \vee (B(aT(acb)b) \wedge B(bT(bac)c)))$.

Let $Ob(x_1x_2x_3)$, $Ac(x_1x_2x_3)$, and $Ri(x_1x_2x_3)$ stand for '$x_1x_2x_3$ is an obtuse, acute, or right triangle respectively'. Their formal definitions are (throughout the paper, the summation in the indices of the x's is done modulo 3, i.e. $x_4 = x_1$ and $x_5 = x_2$; we have abbreviated $\sigma(x_ix_{i+1})$ to σ_i):

$$Ob(x_ix_{i+1}x_{i+2}) \overset{\text{def}}{\leftrightarrow} \bigvee_{i=1}^{3} (j(x_ix_{i+1}x_{i+2}) \wedge B(x_{i+1}T(x_{i+1}T(x_ix_{i+2}H(x_{i+1}x_i\sigma_i))x_{i+2})x_{i+2})$$
$$\wedge \neg(x_{i+2} = T(x_{i+1}T(x_ix_{i+2}H(x_{i+1}x_i\sigma_i))x_{i+2}))),$$

$$Ac(x_ix_{i+1}x_{i+2}) \overset{\text{def}}{\leftrightarrow} \bigvee_{i=1}^{3} (j(x_ix_{i+1}x_{i+2}) \wedge B(x_{i+1}x_{i+2}T(x_{i+1}T(x_ix_{i+2}H(x_{i+1}x_i\sigma_i))x_{i+2}))$$
$$\wedge \neg(x_{i+2} = T(x_{i+1}T(x_ix_{i+2}H(x_{i+1}x_i\sigma_i))x_{i+2}))),$$

$$Ri(x_ix_{i+1}x_{i+2}) \overset{\text{def}}{\leftrightarrow} \neg L(x_ix_{i+1}x_{i+2}) \wedge \bigvee_{i=1}^{3} (R'(x_ix_{i+1}x_{i+2}) = \sigma(x_ix_{i+2})).$$

Let $S_0(abcd) \overset{\text{def}}{=} S(abc\sigma(cd))$ and
$h = h(abcd) \overset{\text{def}}{=} H(S_0(abcd)cd)$.

For three given segments ab, cd and ef, of lengths x, y and z respectively, we denote by $s = s(abcdef)$ the point at distance $\frac{xy}{z}$ from a_0 on ray $a_0\vec{a}_1$. Its definition will be an $\mathrm{L}_{\omega_1\omega}$-formula, which is easily seen to be a halting formula of a program, hence in $aL(\mathrm{L})$. Let:
$l_1(x_0x_1) \overset{\text{def}}{=} x_1$, and $l_n(x_0x_1) \overset{\text{def}}{=} S_0(x_0x_1l_{n-1}(x_0x_1)x_0)$, for $n \geq 2$.

For two given segments ab and cd, with lengths x and y, we denote by $X(abcd)$, $Y(abcd)$, $Z(abcd)$ the points p on the ray $a_0\vec{a}_1$ for which the length of a_0p is $x+y$, respectively $|x-y|$, respectively $\sqrt{xy}$. Their definitions are:
$X(abcd) \stackrel{\text{def}}{=} S(S_0(abcd)da_0a_1)$,
$Y(abcd) \stackrel{\text{def}}{=} S(dS(abcd)a_0a_1)$,
$Z(abcd) \stackrel{\text{def}}{=} S(cha_0a_1)$.
Let, for $n \geq 1$:
$\chi_n = \chi_n(abcdef) \stackrel{\text{def}}{=} Z(a_0X(chel_n(ef))a_0Y(chel_n(ef)))$,
$i_n = i_n(abcdef) \stackrel{\text{def}}{=} S(a_0\chi_n cH(\sigma(ch)ch))$,
$j_n = j_n(abcdef) \stackrel{\text{def}}{=} S(hF_0(hi_nc)a_0a_1)$.
The definition of $s(abcdef)$ can now be stated as:

$$a \neq b \wedge c \neq d \wedge e \neq f \wedge s(abcdef) = g \stackrel{\text{def}}{\leftrightarrow} \bigvee_{n=1}^{\infty} (g = l_n(a_0j_n) \wedge B(eS(chef)l_n(ef))).$$

Let $p(abc)$ be the point on the ray $a_0\vec{a}_1$, whose distance from a_0 is $\frac{x+y+z}{2}$, where x, y, z are the lengths of the segments ab, bc, ca respectively. Its definition is:
$p = p(abc) \stackrel{\text{def}}{=} S(T'(abc)M(T'(abc)T'(cba))a_0a_1)$.
For non-collinear points a, b, c, let $Rad(abc)$ denote the point d on the ray $a_0\vec{a}_1$ for which the length of the segment a_0d is equal to the length of the radius of the circumcircle of the triangle abc. Its definition is:
$\neg L(abc) \wedge Rad(abc) = d \stackrel{\text{def}}{\leftrightarrow} \neg L(abc)$
$\wedge\, d = s(a_0s(cM(ac)cM(bc)a_0Z(a_0pa_0Y(a_0pbc)))aba_0Z(a_0Y(a_0pac)a_0Y(a_0pab)))$.
Let $Or(abca'b'c')$ mean that ab, ac and $a'b', a'c'$ are two pairs of perpendicular segments such that $(\vec{ab}, \vec{ac})$ and $(\vec{a'b'}, \vec{a'c'})$ have the same orientation. With $Hl(xyz)$ standing again for $B(xyz) \vee B(xzy)$, its definition is:

$$\begin{aligned} Or(abca'b'c') \quad \stackrel{\text{def}}{\leftrightarrow} \quad & a \neq b \wedge a \neq c \wedge a' \neq b' \wedge a' \neq c' \wedge R'(acb) = \sigma(ab) \wedge R'(a'c'b') \\ & = \sigma(a'b') \wedge cP(a'ac') \equiv T(acb)P(a'aT(a'c'b')) \wedge \neg(Hl(abP(a'ac)) \\ & \wedge Hl(acP(a'ab')) \wedge \neg(B(baP(a'ac)) \wedge B(caP(a'ab'))). \end{aligned}$$

We can now finally define the operation U, providing us with the centre of the circumcircle of a $\triangle abc$, provided that a, b, c are not collinear. The definition is (where we have denoted by m the point $M(x_{i+1}x_{i+2})$ and by r the point $Rad(x_ix_{i+1}x_{i+2})$):
$\neg L(x_1x_2x_3) \wedge U(x_1x_2x_3) = y \stackrel{\text{def}}{\leftrightarrow} [Ri(x_1x_2x_3) \wedge (\bigvee_{i=1}^{3}(j(x_ix_{i+1}x_{i+2})$
$\wedge\, y = m))] \vee [Ac(x_1x_2x_3) \wedge (\bigvee_{i=1}^{3}(j(x_ix_{i+1}x_{i+2})$
$\wedge\, ((Or(mH(x_{i+1}mx_{i+2})x_{i+1}F_0(x_{i+1}x_{i+2}x_i)x_ix_{i+1})$
$\wedge\, y = S(a_0Z(a_0Y(a_0rx_{i+1}m)a_0X(a_0rx_{i+1}m))mH(x_{i+1}mx_{i+2})))$
$\vee\, (Or(mH(x_{i+2}mx_{i+1})x_{i+1}F_0(x_{i+1}x_{i+2}x_i)x_ix_{i+1})$
$\wedge\, y = S(a_0Z(a_0Y(a_0rx_{i+1}m)a_0X(a_0rx_{i+1}m))mH(x_{i+2}mx_{i+1}))))))]$
$\vee\, [Ob(x_1x_2x_3) \wedge (\bigvee_{i=1}^{3}(j(x_ix_{i+1}x_{i+2})$
$\wedge\, ((Or(mH(x_{i+1}mx_{i+2})x_{i+1}F_0(x_{i+1}x_{i+2}x_i)x_ix_{i+1})$
$\wedge\, y = S(a_0Z(a_0Y(a_0rx_{i+1}m)a_0X(a_0rx_{i+1}m))mH(x_{i+2}mx_{i+1})))$

$\vee\, (Or(mH(x_{i+2}mx_{i+1})x_{i+1}F_0(x_{i+1}x_{i+2}x_i)x_ix_{i+1})$
$\wedge\, y = S(a_0Z(a_0Y(a_0rx_{i+1})ma_0X(a_0rx_{i+1}m))mH(x_{i+1}mx_{i+2}))))))]$.

4.3.4 The axiom system

We have just shown how to eliminate the operations R' and U from the axiom system for $\mathcal{CE}'_2$ given in §3.7.9. Hence all the $\mathrm{L}_{e'}$-axioms for $\mathcal{CE}'_2$ can be thought of as universal axioms in $\mathrm{L}_{\omega_1\omega}$ (i.e. expressed only with a_0, a_1, a_2, T' and H); that axiom system with its axioms written as universal $\mathrm{L}_{\omega_1\omega}$-sentences will be referred to as Σ_0. Moreover, all the axioms are in $aL(\mathrm{L})$, as can easily be checked.

Let *Arch* be the Archimedean axiom, which is also a sentence in $aL(\mathrm{L})$. It can be expressed as:

$$Arch. \quad x \neq y \to \bigvee_{n=1}^{\infty} B(xyz_n(xy)),$$

where $z_1(xy) \overset{\text{def}}{=} S(a_0a_1xy)$, $z_n(xy) \overset{\text{def}}{=} S(a_0a_1z_{n-1}(xy)y)$ for $n \geq 2$.
Let $\Sigma' = \Sigma_0 \cup \{Arch\}$, where *Arch* is also thought of as an axiom in $aL(\mathrm{L})$ (cf. [73, p. 51] for a program Π for which *Arch* is ϕ_Π).

The representation theorem in §3.7.9 and the Theorems 6.39 and 7.5 from [73, p. 51, 56] imply the following

Representation Theorem 4.3.1 *(1) $\mathcal{M} \in Mod(\Sigma')$ if and only if $\mathcal{M} \simeq \mathfrak{E}(\mathfrak{K})$, where K is an Archimedean ordered Euclidean field.*
(2) $aTh(\mathfrak{E}(\mathbb{R})) = aCn(\Sigma')$.

Proof. (1) The Representation Theorem 3.7.4 asserts that every model of $\mathcal{CE}'_2$ is isomorphic to a constructive Cartesian plane over a Euclidean ordered field, with R', U and H having the intended interpretation. This also gives the intended interpretation for T and σ considered as defined notions (from a_0, a_1, a_2, R', U and H). Since $c = \sigma(a\sigma(ac)) \in aCn(\Sigma')$, we have
$\mathbf{T}'_F(\mathbf{abc}) = \mathbf{T}'_F(\mathbf{ab}\sigma_F(\mathbf{a}\sigma_F(\mathbf{ac}))) = \mathbf{T}_F(\mathbf{ab}\sigma_F(\mathbf{ac}))$,
wherefrom we obtain the intended interpretation for T'.

Hence, if $\mathcal{M}$ is a model of Σ', then $\mathcal{M}$ has to be isomorphic to some $\mathfrak{E}(\mathfrak{K})$, where K is an Archimedean ordered Euclidean field. The fact that the $\mathfrak{E}(\mathfrak{K})$'s, with K an Archimedean ordered Euclidean field are models of Σ' follows from the fact that the definitions of T, M, σ, R, B, D, U are valid in a Cartesian plane over K , i.e. that each axiom of Σ' holds in $\mathfrak{E}(K)$.

(2) follows from theorems 6.39 and 7.5 from [73, p.51,56]. □

It is obvious that both operations (T' and H) are needed in order to quantifier-free axiomatize plane Euclidean geometry over Archimedean ordered Euclidean fields, for the closure of $\{\mathbf{a}_0 = (0,0), \mathbf{a}_1 = (1,0), \mathbf{a}_2 = (0,1)\}$ under $\mathbf{T}'_{\mathbb{R}}$ is included in the Pythagorean closure of $\mathbb{Q}$, which is not a Euclidean field, and $\mathbf{H}_{\mathbb{R}}$ applied to the same $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$ does not need to produce any new point since $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ are not collinear.

Moreover, it is not possible to axiomatize plane Euclidean geometry over Archimedean ordered fields by using only a finite number of individual constants and binary geometric operations (we mean here binary operations that are invariant under the group of orientation-preserving similarities)[3], as shown in [22, p. 77-80], so the use of ternary geometric operations is necessary.

However, if the notion of a *geometric* operation is interpreted to mean invarinace under the group of orientation-preserving isometries, then it becomes possible to axiomatize in $aL(\mathrm{L}_2)$ plane Euclidean geometry (with a unit length) over Archimedean ordered Euclidean fields, in a language $\mathrm{L}_2 = \mathrm{L}_2(a_0, a_1, a_2, IUC)$, that contains only three individual constants and a binary operation IUC, which has the intuitive interpretation $IUC(xy) = z$ iff 'z is the intersection point of the circles with centres x and y and radii congruent to a_0a_1 (i. e. of unit length) (that intersection point for which $(\overrightarrow{yz}, \overrightarrow{yx})$ has the same orientation as $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2})$ if the circles intersect in two points), provided that the two circles intersect, and $z = x$ otherwise'. This follows from the results proved in [94].

4.4 Ruler and gauge constructions

We have seen in the previous paragraph that local operations can serve as primitive notions for axiomatizing in algorithmic logic Cartesian planes over Archimedean ordered Euclidean fields.

One naturally would like to know if the same is true for Cartesian planes over Archimedean ordered Pythagorean fields. As F. Bachmann puts it in [8], can all the points of a Cartesian plane over an Archimedean ordered Pythagorean field, that can be constructed by ruler and gauge, also be constructed by gauge, set square and with a ruler, which can be used only to join two already constructed points, but not to construct the intersection point of two intersecting lines? The set square can be used both to construct the footpoint of the perpendicular from a point to a line not containing the point, and for constructing the perpendicular at a point of a given line to that line.

The answer is yes. To see this, let Δ be the axiom system in $aL(\mathrm{L}_{fm})$, obtained by adding the Archimedean axiom *Arch* to the L_{fm}-axiom system for $\mathcal{FM}$ given in §3.7.8. Then, according to Representation Theorem 3.7.3, we have

Representation Theorem 4.4.1 *(1) $\mathcal{M} \in Mod(\Delta)$ iff $\mathcal{M} \simeq \langle K \times K, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{T}_F, \mathbf{M}_F\rangle$, where K is an Archimedean ordered Pythagorean field.*

(2) $aTh(\langle \mathbb{R} \times \mathbb{R}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{T}_F, \mathbf{M}_F\rangle) = aCn(\Delta)$.

This means that the operation U is algorithmically definable from a_0, a_1, a_2, T and M. To see this note that if $K = Pyth(\mathbb{Q}(x_1, x_2, y_1, y_2, z_1, z_2))$, where x_i, y_i, z_i, for $i = 1, 2$ are six independent transcendentals, and $K \subset \mathbb{R}$ with the induced order, then the above Representation Theorem tells us that $\mathbf{U}_K(\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2)$ is in the closure of $\mathbf{a}_0 = (x_1, x_2), \mathbf{a}_1 = (y_1, y_2), \mathbf{a}_2 = (z_1, z_2)$ under the operations T and M. However, we do not know how to actually define U in terms of a_0, a_1, a_2, T and M.

[3] See the comment made after Definition 1.2.3.

4.5 Minimal models, finite models

One of the powerful features of algorithmic logic is that it allows us to axiomatize up to isomorphism certain denumerable structures, like for example the standard model of arithmetic, ω.

We thus ask the natural question: Which geometric structures can be axiomatized up to isomorphism by algorithmic axiom systems? What non-elementary classes of models, besides the Euclidean geometries over Archimedean fields, can be axiomatized inside algorithmic logic?

We shall see that algorithmic logic allows us to characterize minimal models of some geometries, as well as the set of all finite models of the purely metric geometries.

The programs, whose halting formula will state that the model is minimal, essentially state that every point in the plane can be reached from the three given initial points a_0, a_1, a_2 by an iterated application of the operations in the language of that theory.

For the finite models the programs will state that the n-th iterate of a certain function of one argument is the identity.

We shall devote this last section of this chapter to see how this can be done and to list some open problems in this context.

4.5.1 Minimal Euclidean and metric-Euclidean planes

As first-order axioms we take the L_{eu}-axioms for $\mathcal{CD}_2'$ (hence a_0, a_1, a_2 will stand for the vertices of a right isosceles triangle, with $\angle a_1a_0a_2$ right). We shall first show how to algorithmically axiomatize $\mathfrak{E}_2(\mathbb{Q},k)$[4] for positive, square-free integers k. We begin with the case $k=1$.

In order to express intelligibly the program whose halting formula will state that every point in the plane can be reached from the given points a_0, a_1, a_2 by some composition of the operations U and R', we need the following abbreviations (see Fig. 4.3):
$y(x) \stackrel{\text{def}}{=} R'(a_0P(a_0a_2a_1)x)$,
$\delta(x,y,z) \stackrel{\text{def}}{=} U(F(xyz)\sigma(zF(xyz))R'(a_0y\sigma(zF(xyz))))$,
having the intuitive interpretations:
$y(x)$ is the point on the line a_0a_2 (which we may think of as the y-axis) whose coordinates are $(0,\alpha)$, whenever the coordinates of x are $(\alpha,0)$, and
$\delta(x,y,z)$ is the point on the line a_0a_1 (which we may think of as the x-axis), whose coordinates are $(\frac{\beta\gamma}{\alpha},0)$ whenever the coordinates of x, y, z are $(0,\alpha)$, $(\beta,0)$ and $(0,\gamma)$ respectively (which will be the only case when this abbreviation will be used). We now introduce the function $f(x,n)$ — which, for $x=a_1$, represents a point with coordinates $(n,0)$, with n a positive integer — by setting $f(x,0) \stackrel{\text{def}}{=} a_0$, $f(x,1) \stackrel{\text{def}}{=} x$, $f(x,n+2) \stackrel{\text{def}}{=} \sigma(f(x,n+1)f(x,n))$ for $n \geq 0$. Let now $n \geq 2$ be an integer and let $z=f(a_1,n)$, $t=f(a_1,m)$, $u=f(a_1,a)$, $v=f(a_1,b)$,

[4]Throughout this section the interpretation of a_0, a_1, a_2 in $\mathfrak{E}_2(\mathbb{Q},k)$ will be considered to be $(0,0)$, $(1,0)$ and $(0,1)$ respectively, as opposed to having $\mathbf{a}_2=(\alpha,\beta)$ with $\alpha,\beta\in\mathbb{Q}$ as in Representation Theorem 3.7.1. The reason for doing so lies mainly in the improved intelligibility of the programs (whose halting formulas describe the minimality of the geometry) achieved by this change.

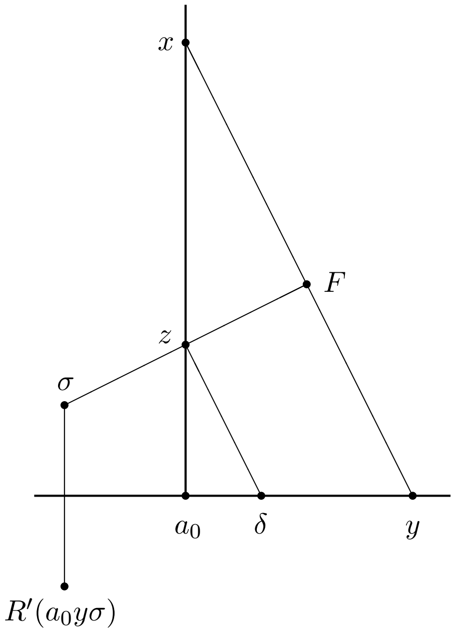

Figure 4.3: The definition of $\delta(x, y, z)$

with

$$0 < a \leq m < b \leq n. \tag{4.1}$$

Let $\alpha \stackrel{\text{def}}{=} y(P(ta_0v))$, $\beta \stackrel{\text{def}}{=} y(P(va_0z))$,

$$\gamma = \gamma(\alpha, \beta) \stackrel{\text{def}}{=} \begin{cases} a_1, & \text{if } \alpha = \beta \\ \delta(\alpha, a_1, \beta), & \text{if } \alpha \neq \beta \wedge (\alpha = a_2 \vee \beta = a_2) \\ \delta(a_2, \delta(\alpha, a_1, a_2), \beta), & \text{if } \alpha \neq \beta \wedge \alpha \neq a_2 \wedge \beta \neq a_2 \end{cases}$$

In the standard interpretation, with $\mathbf{a}_0 = (0, 0)$, $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (0, 1)$, we have $\mathbf{z} = (n, 0)$, $\mathbf{t} = (m, 0)$, $\mathbf{u} = (a, 0)$, $\mathbf{v} = (b, 0)$, $\alpha = (0, b - m)$, $\beta = (0, n - b)$, $\gamma = (\frac{n-b}{b-m}, 0)$

Let $\alpha' \stackrel{\text{def}}{=} y(u)$, $\beta' \stackrel{\text{def}}{=} y(P(a_0ut))$, $\gamma' \stackrel{\text{def}}{=} \gamma(\alpha', \beta')$, i. e., in the standard interpretation, $\alpha' = (0, a)$, $\beta' = (0, m - a)$, $\gamma' = (\frac{m-a}{a}, 0)$.

Let $w_1 = P(a_0\gamma' y(\gamma))$, $w_2 = R'(a_0a_1w_1)$, $w_3 = R'(a_0a_2w_2)$, $w_4 = R'(a_0a_1w_3)$, i. e. $\mathbf{w}_1 = (\frac{m-a}{a}, \frac{n-b}{b-m})$, $\mathbf{w}_2 = (\frac{m-a}{a}, -\frac{n-b}{b-m})$, $\mathbf{w}_3 = (-\frac{m-a}{a}, -\frac{n-b}{b-m})$, $\mathbf{w}_4 = (-\frac{m-a}{a}, \frac{n-b}{b-m})$.

Now the flowchart will, for some given input x, check, looping over n, m, a and b, whether that x is equal to either a_0 or some w_i for some $i = 1, 2, 3, 4$. The halting formula will thus have the following form:

$$x = a_0 \vee (\bigvee_{n,m,a,b} x = w_1 \vee x = w_2 \vee x = w_3 \vee x = w_4)$$

where $\bigvee_{n,m,a,b}$ stands for the infinite disjunction over all $n \in \mathbb{N}$ and all m, a, b that satisfy (4.1).

The L_{eu}-axiom system for $\mathcal{CD}_2'$, together with axioms stating that $f(a_1, n) \neq a_0$ for all $n \geq 1$, and with the above halting formula thus represents an axiom system in $aL(L_{eu})$ for $aTh(\mathfrak{E}_2(\mathbb{Q}, 1))$, which has, up to isomorphism a unique model, namely $\mathfrak{E}_2(\mathbb{Q}, 1)$.

We shall see that $aTh(\mathfrak{E}_2(\mathbb{Q}, k))$, with $k \geq 1$ a squarefree integer, can be axiomatized with only slight changes (in the definition of $y(x)$ and in the axiom A3.5.2). We first note that we get an isomorphic copy of $\mathfrak{E}_2(\mathbb{Q}, k)$ if we map all points (x, y) with $x, y \in \mathbb{Q}$ into the points $(x, y\sqrt{k})$, considered as points in $\mathfrak{E}_2(\mathbb{Q}(\sqrt{k}), 1)$, i. e. the map $\varphi : \mathfrak{E}_2(\mathbb{Q}, k) \to \mathfrak{E}_2(\mathbb{Q}(\sqrt{k}), 1)$ defined by $\varphi(x, y) = (x, y\sqrt{k})$ is an isomorphism onto its image (as observed in [7, §3]). We shall henceforth think of $\varphi(\mathfrak{E}_2(\mathbb{Q}, k))$ as the 'standard model', hence the three noncollinear points a_0, a_1, a_2 will have the standard interpretation of $\mathbf{a}_0 = (0, 0)$, $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (0, \sqrt{k})$. Let (x_k, y_k) be a positive integer solution (say the minimal positive solution) of PELL's equation $x^2 - ky^2 = 1$. We need to have an axiom that states that the orthogonality constant is k, or equivalently that the interpretation of a_0, a_1, a_2 in $\varphi(\mathfrak{E}_2(\mathbb{Q}, k))$ is standard. With the abbreviation f introduced above, the axiom can be stated as

A 4.5.1 (k) $a_0 \neq a_1 \wedge a_0 \neq a_2 \wedge a_1 \neq a_2 \wedge R'(a_0a_1a_2) = \sigma(a_0a_2) \wedge a_1 f(a_2, y_k) \equiv a_0 f(a_1, x_k)$.

Let $\mathcal{CD}_2(k)$ be the theory having as axioms those of $\mathcal{CD}_2$ together with A4.5.1(k) (A3.3.11 may be omitted, as it is implied by A4.5.1(k)).

The only change needed in the informal description of the program given above for the case $k = 1$ is in the definition of $y(x)$. It now is

$y_k(x) \stackrel{\text{def}}{=} f(R'(a_0P(a_0a_2a_1)x), k)$

and its standard interpretation, for $\mathbf{x} = (\xi, 0)$, is $\mathbf{y}_k(\mathbf{x}) = (0, \xi\sqrt{k})$. To sum up, if we add the halting formula for the modified program to the axiom system for $\mathcal{CD}_2(k)$, we get an axiom system in $aL(L_{eu})$ for $aTh(\mathfrak{E}_2(\mathbb{Q}, k))$, which has, up to isomorphism a unique model, namely $\mathfrak{E}_2(\mathbb{Q}, k)$.

We also see for the first time that an orthogonality constant $k \neq 1$ can appear quite naturally, for if one takes three points $\mathbf{a}_0 = (0, 0)$, $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (0, \sqrt{k})$ in $\mathfrak{E}_2(\mathbb{Q}(\sqrt{k}), 1)$ (i. e. in an Euclidean plane in the classical sense — we may indeed think of these three points as being in $\mathfrak{E}_2(\mathbb{R}, 1)$ as well) and closes under the operations R' and U, one gets an isomorphic copy of a Euclidean geometry with orthogonality constant k over $\mathbb{Q}$.[5]

One can similarly algorithmically axiomatize the minimal metric-Euclidean plane of characteristic zero and with orthogonality constant 1. The algebraic characterization of it was provided by F. BACHMANN in [5, §3] and [10, §19,2]. The minimal metric-Euclidean plane of characteristic 0 with orthogonality constant 1 is isomorphic to $\langle \mathfrak{C}(\mathbb{Q}) \times \mathfrak{C}(\mathbb{Q}), (0, 0), (1, 0), (0, 1), \mathbf{P}_{\mathbb{Q}}, \mathbf{F}_{(\mathbb{Q}, 1)} \rangle$, where $\mathfrak{C}(\mathbb{Q})$ is the subring of $\mathbb{Q}$ consisting of all the rational numbers whose denominators are of the form $m^2 + n^2$, for some $m, n \in \mathbb{N}$ (or equivalently the subring of $\mathbb{Q}$ consisting of all integers and all irreducible fractions whose denominators contain only primes of the form $4m + 1$ and 2 as prime factors).

The axiom system for the minimal metric-Euclidean plane of characteristic 0 and orthogonality constant 1 will contain all the axioms for $\mathcal{ME}'$, axioms stating that the characteristic of the coordinate field is 0, and a halting-formula for a flow-chart program that will 'say' that an arbitrary point is in the closure of the three given points a_0, a_1, a_2 under the opeartions R and P.

[5] This fact has been first noted by F. BACHMANN in [7, §4].

In order to uniformly generate all the points in this minimal metric-Euclidean plane, starting from the three given points a_0, a_1, a_2, one notes that the footpoint of the perpendicular from $(1,0)$ to the line $y = cx$ has the coordinates $(\frac{1}{1+c^2}, \frac{c}{1+c^2})$. This means that the footpoint of the perpendicular from it to the x-axis has the coordinates $(\frac{1}{1+c^2}, 0)$ hence whenever there are two points from our minimal plane that lie on the line $y = cx$, the point $(\frac{1}{1+c^2}, 0)$ will be there, constructed by a double use of the F operation. Let m and n be positive integers and r and s be integers. We know how to construct, starting from the three given points, two points on the line $y = \frac{m}{n}x$ (one being a_0 and the other one being $P(a_0 f(a_1, n) f(a_2, m)))$. Hence we know how to get the point $(\frac{n^2}{m^2+n^2}, 0)$. By symmetry, we know how to get $(\frac{m^2}{m^2+n^2}, 0)$. We hence know how to construct $(r\frac{m^2}{m^2+n^2} + s\frac{n^2}{m^2+n^2}, 0)$ (since we know how to add and subtract segments — using P or σ). If we let the program loop over m, n, r, s it will generate all the points in $\mathfrak{C}(\mathbb{Q})$ on the x-axis. It is now easy to see that all points in $\mathfrak{C}(\mathbb{Q}) \times \mathfrak{C}(\mathbb{Q})$ can be generated by a flow-chart program. We also need to add the axioms $f(a_1, n) \neq a_0$ for all $n \geq 2$ to the axiom system. It is easy to see that this minimal metric-Euclidean plane is not a Euclidean plane, so the Euclidean operation U would be needed even for characterizing the minimal Euclidean plane.

Note that in expressing the halting formula for the above program we use in fact only one variable.

4.5.2 Finite Euclidean planes

We finally turn to the algorithmic axiomatization of the set of all finite metric-Euclidean planes and the set of all finite rectangular planes (it will turn out that there is only a language-difference between the two, since both are Euclidean planes, as proved for the metric-Euclidean case by F. Bachmann in [10] (in fact all finite metric planes are Euclidean), and for the rectangular case by Stanik in [79, p. 4]).

Since a_0, a_1, a_2 are non-collinear, one of a_2 or $P(a_0a_1a_2)$ will not lie on the perpendicular at a_0 to a_0a_1. Let p be a_2 if $\neg\, a_0a_2 \perp a_0a_1$, and $P(a_0a_1a_2)$ if $a_0a_2 \perp a_0a_1$.

In order to axiomatize in $aL(\mathrm{L}_{me})$ the set of all finite metric-Euclidean planes, we need, besides the axiom system for $\mathcal{CME}$, a halting formula for a program that runs as follows:
Given a point x, not on the line a_0a_1 and not on the perpendicular at a_0 to a_0a_1, it halts iff the first time $r_n(p) = a_1$ (i. e. for the first n for which $r_n(p) = a_1$) we also have $r_n(x) = a_1$, where $r_1(x) \stackrel{\text{def}}{=} a_1$, $q_n(x) \stackrel{\text{def}}{=} F(a_0 x r_n)$, $r_{n+1}(x) \stackrel{\text{def}}{=} F(a_0 x q_n)$, for $n \geq 1$.
To better understand why this halting formula really implies that the plane is finite, we analyze the statement of the halting formula for the case in which the orthogonality constant $k = 1$.[6] If a_0, a_1 have the standard interpretation $(0,0)$ and $(0,1)$ respectively, and the equation of the line a_0x is $y = cx$, then the points $r_n(x)$ have coordinates $\frac{1}{(1+c^2)^n}$ and $r_n(x) = a_1$ implies $(1+c^2)^n = 1$. Therefore, the halting formula states that, for that $n \in \mathbb{N}$, for which $(1+s^2)^n = 1$ and $(1+s^2)^m \neq 1$ for $1 \leq m < n$, where s is the slope of the line a_0p, every element of the field is a solution of the equation $(1+X^2)^n - 1 = 0$.

[6] Notice that in finite Euclidean planes the orthogonality constant can always be normalized to 1, since in a finite field of characteristic $\neq 2$ every nonzero element is either a square or the opposite of a square (since the map $\varphi : K^* \to K^*$, defined by $\varphi(x) = x^2$ is two to one, K^* being $K \setminus \{0\}$). The same will apply to finite metric-Euclidean or rectangular planes, since these are Euclidean planes.

In order to axiomatize in $aL(\mathrm{L}_{re})$ the set of all finite rectangular planes, we need, besides the axiom system for $\mathcal{CRE}$, a halting formula for the same program as in the metric-Euclidean case, with F replaced by R everywhere. The resulting equation, that all elements of the field have to satisfy, will no longer be as simple as in the metric-Euclidean case, but this does not matter. A field in which all elements are solutions of a polynomial equation is finite.

It thus follows that in the finite case, just as in the Archimedean case, we need no specifically Euclidean, non-local operation (like U), in order to axiomatize Euclidean planes. They can be axiomatized in both $aL(\mathrm{L}_{me})$ and $aL(\mathrm{L}_{re})$. To put it differently, U is algorithmically definable from R and P (or from F and P).

4.6 The algorithmic theory of the real Euclidean plane

We finally intend to determine what the algorithmic theories of planes over the field of real numbers are.

Let Φ be a quantifier-free axiom system for the theory of formally real fields in the language $\mathrm{L}_f = \mathrm{L}(+,\cdot,-,^{-1},0,1)$ and Ψ be an axiom system for Archimedean ordered fields in the algorithmic language $aL(\mathrm{L}_{of})$, where $\mathrm{L}_{of} = \mathrm{L}(\leq,+,\cdot,-,^{-1},0,1)$; let $\mathbb{R}_= = \langle\mathbb{R},+,\cdot,-,^{-1},0,1\rangle$ and $\mathbb{R}_\leq = \langle\mathbb{R},\leq,+,\cdot,-,^{-1},0,1\rangle$. Engeler [20] has proved that (cf. [73, Satz 2.1 and Satz 2.3])

$$aTh(\mathbb{R}_=) = aCn(\Phi) \text{ and } aTh(\mathbb{R}_\leq) = aCn(\Psi). \tag{4.2}$$

We denote by $as(\mathcal{T})$ a quantifier-free axiom system for the theory $\mathcal{T}$. For $\mathcal{T}$ one of $\mathcal{CD}_2$, $\mathcal{CD}'_2$ or $\mathcal{CBD}'_2$ the corresponding axiom systems can be found in §3.7.4 and §3.7.5. Let α, β be two independent transcendentals (i. e. α is transcendental over $\mathbb{Q}$ and β is transcendental over $\mathbb{Q}(\alpha)$). From (4.2) we deduce the following

Theorem 4.6.1
(i) $aTh(\langle\mathbb{R}\times\mathbb{R},\mathbf{R}'_{(\mathbb{R},1)},\mathbf{U}_{(\mathbb{R},1)},(0,0),(1,0),(\alpha,\beta)\rangle) = aCn(as(\mathcal{CD}_2),\{\tau_n|n\in\mathbb{N}\})$,[7]
(ii) $aTh(\mathfrak{E}_2(\mathbb{R},\imath)) = \mathfrak{aCn}(\mathfrak{as}(\mathcal{CD}'_2),\{\tau_\mathfrak{n}|\mathfrak{n}\in\mathbb{N}\})$,
(iii) $aTh(\langle\mathbb{R}\times\mathbb{R},\mathbf{R}'_{(\mathbb{R},1)},\mathbf{U}_{(\mathbb{R},1)},\mathbf{O}_{B\mathbb{R}},(0,0),(1,0),(\alpha,\beta)\rangle) = aCn(as(\mathcal{CBD}'_2),Arch)$.[8]

For the proof of (i) notice that the three points $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ were chosen so that they are 'in general position', by which we mean the following: Whenever we have three non-collinear points $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ in $\mathbb{R}\times\mathbb{R}$, we can, by a change of coordinates, get $\mathbf{a}_0=(0,0)$ and $\mathbf{a}_1=(1,0)$; the third point's coordinates were chosen such that any positive sentence φ (i. e. φ contains no negation sign) in $aL(\mathrm{L}_{eu})$, in which R' is applied only to triples (x,y,z) with $x\neq y$ and U is applied only to triples (x,y,z) of noncollinear points, which is true (in the model $\langle\mathbb{R}\times\mathbb{R},(0,0),(1,0),(\alpha,\beta),\mathbf{R}'_{(\mathbb{R},1)},\mathbf{U}_{(\mathbb{R},1)}\rangle$) about this particular choice of a non-collinear triple $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, would have to be true about any other three non-collinear points, and hence belongs to the right hand side of (i). To see this, consider φ^*, the algebraic translation of φ in L_f. Each disjunct of the disjunctive normal form of φ^* will have to be a system of equalities in α, β and (possibly) a finite number of indeterminates, with integer coefficients. Since α, β are independent transcendentals, for the purpose of establishing the validity of any equation

[7] τ_n are here the sentences introduced in §3.8.
[8] *Arch* stands for the Archimedean axiom as stated in §4.3.4.

they can be thought of as indeterminates, and so these equalities will have to be true for all values of α, β, hence φ has to be valid for all non-collinear $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$.

Part (i) of this theorem shows that, far from being artificial, the theory $\mathcal{CD}_2$ (which is the theory of constructive Euclidean planes with arbitrary orthogonality constant k (whose opposite is not a square)) is a 'naturally occuring' structure. It is an exact statement of the vague impression conveyed by [69], where the author develops elementary geometry using Euclidean planes over arbitary fields of charactersitic $\neq 2$ with an arbitrary orthogonality constant, that almost all theorems of plane Euclidean geometry that involve only metric notions are true in all Euclidean planes over any field of characteristic $\neq 2$ with arbitrary orthogonality constant. If one thinks of the τ_n's as not particularly interesting geometric theorems, then the 'geometric content' of the constructive Euclidean plane over the reals, with $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ in general position, would be just $\mathcal{CD}_2$.

æ

Appendix A

Index of axioms

A1.1.1 $ab \equiv ba$
A1.1.2 $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$
A1.1.3 $ab \equiv cc \rightarrow a = b$
A1.1.4 $(\forall abcq)(\exists x)\,[B(qax) \wedge ax \equiv bc]$
A1.1.5 $a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd'$
$\rightarrow cd \equiv c'd'$
A1.1.6 $B(aba) \rightarrow a = b$
A1.1.7 $(\forall abcpq)(\exists x)\,[B(apc) \wedge B(bqc) \rightarrow B(pxb) \wedge B(qxa)]$
A1.1.8 $(\exists abc)\neg L(abc)$
A1.1.9 $p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \rightarrow L(abc)$
A1.1.10 $(\forall abcdt)(\exists xy)\,[B(adt) \wedge B(bdc) \wedge a \neq d \rightarrow B(abx) \wedge B(acy) \wedge B(xty)]$
A1.1.11 [Continuity axiom schema] $(\exists a)(\forall xy)\,[\alpha(x) \wedge \beta(y) \rightarrow B(axy)]$
$\rightarrow (\exists b)(\forall xy)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)]$
where $\alpha(x)$, $\beta(y)$ are formulas in $\mathrm{L}_{B\equiv}$
with a, b, y not occuring free in $\alpha(x)$ and a, b, x not occuring free in $\beta(y)$.
A1.1.12 $(\forall abcpqr)(\exists x)\,[B(cqp) \wedge B(cpr) \wedge ca \equiv cq \wedge cb \equiv cr \rightarrow cx \equiv cp \wedge B(axb)]$
A2.1.1 $ab \equiv cd \rightarrow cd \equiv ab$
A2.1.2 $ab \equiv cd \wedge cd \equiv ef \rightarrow ab \equiv ef$
A2.1.3 $(\forall abca'b'c'xm)(\exists x')(\forall y)\,[a \neq b \wedge c \neq m \wedge ac \equiv am \wedge bc \equiv bm \wedge a'b' \equiv ab$
$\wedge\, a'c' \equiv ac \wedge b'c' \equiv bc \rightarrow a'x' \equiv ax \wedge b'x' \equiv bx \wedge c'x' \equiv cx$
$\wedge\, (a'y \equiv ax \wedge b'y \equiv bx \wedge c'y \equiv cx \rightarrow y = x')]$
A2.1.4 $(\forall abcd)(\exists m)\,[a \neq b \wedge c \neq d \wedge ac \equiv ad \wedge bc \equiv bd \rightarrow am \equiv cm \wedge bm \equiv cm$
$\wedge\, m \neq a \wedge m \neq c]$
A2.1.5 $(\exists abcde)\,[a \neq c \wedge b \neq d \wedge ab \equiv bc \wedge bc \equiv cd \wedge cd \equiv da \wedge ae \equiv be$
$\wedge\, ce \equiv be \wedge ce \equiv de]$
A2.1.6 $(\forall abc)(\exists uv)\,[B(abc) \rightarrow u \neq v \wedge au \equiv av \wedge bu \equiv bv \wedge cu \equiv cv]$
A2.1.7 $B(abc) \rightarrow B(cba)$
A2.1.8 $B(abd) \wedge B(bcd) \rightarrow B(abc)$
A2.1.9 $(\forall abcdp)(\exists q)\,[B(apd) \wedge B(bdc) \rightarrow L(bpq) \wedge L(aqc)]$
A2.1.10 $(\exists abcd)(\forall m)(\exists s)\,[a \neq b \wedge ab \not\equiv cd \wedge (cm \equiv dm \rightarrow ab \equiv cs \wedge cm \equiv sm)]$

A2.1.11 $(\forall abcde)(\exists fghf'g'h')\,[d \neq a \wedge a \neq c \wedge b \neq a \wedge B(bac) \wedge B(eac) \wedge ae \equiv ac$
$\wedge\, de \equiv dc \rightarrow B(daf) \wedge B(daf') \wedge B(bfh) \wedge B(bf'h') \wedge fb \equiv fh \wedge f'b \equiv f'h'$
$\wedge\, gb \equiv gh \wedge g'b \equiv g'h' \wedge B(agc) \wedge B(ag'c) \wedge ag' \equiv gc]$

A2.1.12 $(\forall apqc)(\exists b)\,[ap \equiv aq \wedge a \neq c \rightarrow bp \equiv bq \wedge ab \equiv ac]$

A2.1.13 $(\exists abcc'm)(\forall m')\,[c \neq c' \wedge ac \equiv ac' \wedge bc \equiv bc' \wedge ma \equiv mb$
$\wedge\, (ma \equiv m'a \wedge mb \equiv m'b \rightarrow m' = m]$

A2.2.1 $a \neq b \wedge ((B(abc) \wedge B(abd)) \vee (B(abc) \wedge B(dab)) \vee (B(bca) \wedge B(bda))) \rightarrow L(acd)$

A2.2.2 $ab \equiv cc \vee cc \equiv ab \rightarrow a = b$

A2.2.3 (i) $ab \equiv cd \wedge cd \equiv ce \rightarrow ab \equiv ce$
(ii) $ab \equiv ac \wedge ac \equiv de \rightarrow ab \equiv de$
(iii) $ab \equiv cd \wedge cd \equiv ae \rightarrow ab \equiv ae$

A2.2.4 $(\forall abc)(\exists d)(\forall e)\,[B(cad) \wedge ab \equiv ad \wedge (a \neq c \wedge B(cae) \wedge ab \equiv ae \rightarrow d = e)]$

A2.2.5 $B(abc) \wedge (B(ade) \vee B(aed)) \wedge ab \equiv ad \wedge ac \equiv ae) \rightarrow B(ade) \wedge bc \equiv de$

A2.2.6 $a \neq b \wedge ac \equiv ad \wedge bc \equiv bd \wedge B(abe) \rightarrow ec \equiv ed$

A2.2.7 $ab \equiv ad \wedge ((B(abc) \wedge B(ade)) \vee (B(cab) \wedge B(ead)) \wedge ac \equiv ae \rightarrow dc \equiv be$

A2.2.8 $(\forall ab)(\exists c)\,[B(acb) \wedge ca \equiv cb]$

A2.2.9 $(\forall abc)(\exists d)\,[\neg L(abc) \rightarrow da \equiv db \wedge db \equiv dc]$

A2.2.10 $(\forall abcd)(\exists e)\,[B(abc) \rightarrow B(dbe) \wedge ae \equiv ac]$

A2.3.1(n) $(\exists a_1 a_2 \ldots a_{n+1})[\bigwedge_{p<q,r<s} a_p a_q \equiv a_r a_s]$

A2.4.1 $B(acb) \wedge B(adb) \wedge ae \equiv ad \wedge be \wedge bc \wedge c \neq d \rightarrow B(acd)$

A2.4.2 $d \neq a \wedge B(bac) \wedge ab \equiv ac \wedge db \equiv dc \wedge ae \equiv bd \wedge (B(ade) \vee B(aed)) \rightarrow B(ade)$

A2.5.1 $(\forall abcxyztu)(\exists d)\,[a \neq b \wedge b \neq c \wedge B(xba) \wedge ba \equiv bx \wedge B(ycb) \wedge cb \equiv cy$
$\wedge\, zb \equiv zy \wedge z \neq c \wedge B(tab) \wedge at \equiv ab \wedge u \neq a \wedge ub \equiv ut \rightarrow L(czd) \wedge L(aud)]$

A2.5.2 $(\forall oxyy'a)(\exists pqq'z)\,[B(y'oy) \wedge oy \equiv oy' \wedge a \neq o \wedge ay' \equiv ay \wedge B(axy)$
$\rightarrow (B(oxp) \vee B(opx)) \wedge (B(oyq) \vee B(oqy)) \wedge B(oqq')$
$\wedge\, qo \equiv qq' \wedge po \equiv pq' \wedge pq \equiv oz \wedge B(oyz)]$

A3.2.1 $L(aba)$

A3.2.2 $L(abc) \rightarrow L(cba) \wedge L(bac)$

A3.2.3 $a \neq b \wedge L(abc) \wedge L(abd) \rightarrow L(acd)$

A3.2.5 $aa \equiv bb$

A3.2.6 $(\forall abca'b')(\exists^{=1}c')\,[a \neq b \wedge L(abc) \wedge ab \equiv a'b' \rightarrow L(a'b'c') \wedge ac \equiv a'c' \wedge bc \equiv b'c']$

A3.2.7 $\neg L(abx) \wedge L(abc) \wedge L(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c'$
$\wedge\, ax \equiv a'x' \wedge bx \equiv b'x' \rightarrow xc \equiv x'c'$

A3.2.8 $(\forall abx)(\exists^{=1}x')\,[\neg L(abx) \rightarrow x' \neq x \wedge ax \equiv ax' \wedge bx \equiv bx']$

A3.2.9 $\neg L(abx) \wedge \neg L(aby) \wedge ax \equiv ax' \wedge bx \equiv bx' \wedge x \neq x' \wedge ay \equiv ay' \wedge by \equiv by'$
$\wedge\, y \neq y' \rightarrow xy \equiv x'y'$

A3.2.10 $(\forall abxx')(\exists y)\,[\neg L(abx) \wedge x' \neq x \wedge ax \equiv ax' \wedge bx \equiv bx' \rightarrow L(aby) \wedge L(xx'y)]$

A3.2.11 $(\forall ab)(\exists^{=1}b')\,[a \neq b \rightarrow L(abb') \wedge ab \equiv ab' \wedge b' \neq b]$

A3.2.12 $(\forall xyzab)(\exists c)\,[x \neq y \wedge y \neq z \wedge z \neq x \wedge L(xyz) \wedge L(xya) \wedge ax \equiv ay \wedge L(xyb)$
$\wedge\, by \equiv bz \rightarrow cz \equiv cx]$

A3.2.13 $\neg L(xyz) \wedge L(axy) \wedge ax \equiv ay \wedge L(bzy) \wedge bz \equiv by \wedge L(cxz) \wedge cx \equiv cz \rightarrow \neg L(abc)$,

A3.2.14 $(\exists a_1 a_2 a_3 a_4 a_1' a_2' a_3' a_4')\,[(\bigwedge_{i\neq j} a_i \neq a_j) \wedge (\bigwedge_{i=1}^{4} a_i \neq a_{i'} \wedge L(a_i a_{i+1} a_{i+1}')$
$\wedge\, a_i a_{i+1} \equiv a_i a_{i+1}' \wedge a_{i-1} a_{i+1} \equiv a_{i-1} a_{i+1}')]$[1]

[1] Addition in the indices is *mod 4* both in A3.2.14 and in A3.2.15.

A3.2.15 $(\exists a_1a_2a_3a_4a_1'a_2'a_3'a_4')\,[(\bigwedge_{i\neq j} a_i \neq a_j) \wedge (\bigwedge_{i=1}^4 a_i \neq a_{i'} \wedge L(a_ia_{i+1}a_{i+1}')$
$\wedge\, a_ia_{i+1} \equiv a_ia_{i+1}' \wedge a_{i-1}a_{i+1} \equiv a_{i-1}a_{i+1}') \wedge a_1a_2 \equiv a_2a_3]$
A3.2.16 $(\forall aba'b')(\exists uvu'v'z)\,[\neg L(aba') \wedge a' \neq b' \to \neg L(abz) \wedge \neg L(a'b'z) \wedge L(abu)$
$\wedge\, L(a'b'u') \wedge L(uzu') \wedge L(abv) \wedge L(a'b'v') \wedge L(vzv') \wedge u \neq v]$
A3.3.1 $P(abc) = P(acb)$
A3.3.2 $P(abc) = c \to a = b$
A3.3.3 $\sigma(ax) = \sigma(bx) \to a = b$
A3.3.4 $a \neq b \wedge c \neq d \wedge F(abc) = c \wedge F(abd) = d \to F(abx) = F(cdx)$,
A3.3.5 $\neg L(abx) \wedge x \neq x' \wedge I(axx') \wedge I(bxx') \to x' = R(abx)$
A3.3.6 $L(ab\sigma(ab))$
A3.3.7 $a \neq b \to xy \equiv R(abx)R(aby)$
A3.3.8 $P(abd) = P(cP(abc)d)$
A3.3.9 $I(oab) \wedge I(obc) \to I(oac)$
A3.3.10 $I(oab) \to I(o'P(oao')P(obo'))$
A3.3.11 $\neg L(a_0a_1a_2)$
A3.3.12 $a_0 \neq a_1 \wedge F(a_0a_1a_2) = a_0 \wedge I(a_0a_1a_2)$.
A3.4.1 $ab\|cd \to a \neq b \wedge c \neq d$
A3.4.2 $c \neq e \wedge ab\|cd \wedge L(cde) \to ab\|ce$,
A3.4.3 $a \neq b \to ab\|ba$
A3.4.4 $ab\|cd \to cd\|ab$
A3.4.5 $ab\|cd \wedge cd\|ef \to ab\|ef$
A3.4.6 (i) $(\forall pxy)(\exists u)\,[x \neq y \to pu\|xy]$,
(ii) $ab\|cd \wedge ab\|ce \to L(cde)$
A3.4.7 $(\forall abc)(\exists d)\,[\neg L(abc) \to ad\|bc \wedge cd\|ab]$
A3.4.8 $(\forall oabc)(\exists uvw)\,[\neg L(oab) \wedge \neg L(oac) \wedge \neg L(ocb) \to u \neq o \wedge v \neq o$
$\wedge\, w \neq o \wedge L(oau) \wedge L(obv) \wedge L(ocw) \wedge L(uvw)]$
A3.4.9 $\neg L(abc) \wedge ab\|cd \to (ab \equiv cd \leftrightarrow ac\|bd \vee ad\|bc)$
A3.4.10 $(\forall abcd)(\exists e)\,[\neg L(abc) \wedge ab \equiv ac \wedge L(abd) \to L(ace) \wedge de\|bc \wedge ad \equiv ae]$
A3.4.11 $L(abc) \wedge \neg L(abd) \wedge ad \equiv ad' \wedge bd \equiv bd' \to cd \equiv cd'$
A3.4.12 $(\exists abcd)\,[\neg L(abc) \wedge ad\|bc \wedge cd\|ba \wedge ac \not\| bd]$
A3.4.13 $(\exists abc)\,\neg L(abc)$
A3.4.14 $\tau_{ab}(a) = b$
A3.4.15 $\tau_{ab}(x) = x \leftrightarrow a = b$
A3.4.16 $p \neq q \to pq\|\tau_{ab}(p)\tau_{ab}(q)$
A3.4.17 $a \neq b \to p\tau_{ab}(p)\|q\tau_{ab}(q)$
A3.4.18 $\tau_{cd}(\tau_{ab}(x)) = \tau_{a\tau_{cd}(b)}(x)$
A3.4.19 $a \neq b \wedge c \neq d \wedge L(abc) \wedge L(abd) \to \rho_{ab}(x) = \rho_{cd}(x)$
A3.4.20 $a \neq b \to (\rho_{ab}(x) = x \leftrightarrow L(abx))$
A3.4.21 $a \neq b \wedge xy\|uv \to \rho_{ab}(x)\rho_{ab}(y)\|\rho_{ab}(u)\rho_{ab}(v)$
A3.4.22 $a \neq b \to \rho_{ab}(\rho_{ab}(x)) = x$
A3.4.23 $(\forall a_1a_2a_3b_1b_2b_3o)(\exists ab)(\forall x)\,[\bigwedge_{i=1}^3 (a_i \neq b_i \wedge L(oa_ib_i)) \to a \neq b \wedge L(oab)$
$\wedge\, \rho_{a_1b_1}(\rho_{a_2b_2}(\rho_{a_3b_3}(x)))) = \rho_{ab}(x)]$.
A3.5.1 $a \neq b \wedge c \neq d \wedge R(abc) = c \wedge R(abd) = d \to R(abx) = R(cdx)$
A3.5.2 $a_0 \neq a_1 \wedge R(a_0a_1a_2) = \sigma(a_0a_2) \wedge I(a_0a_1a_2)$

A3.7.1 $a \neq c \wedge a \neq b \wedge F(abc) = c \rightarrow F(abx) = F(acx)$
A3.7.2 $F(abx) = F(bax)$
A3.7.3 $a \neq c \wedge a \neq b \wedge R(abc) = c \rightarrow R(abx) = R(acx)$
A3.7.4 $R(abx) = R(bax)$
A3.7.5 $\neg L(abc) \rightarrow I(U(abc)ab) \wedge I(U(abc)bc)$
A3.7.6 $o \neq a \wedge a \neq b \wedge B(oab) \wedge L(oa'b') \wedge aa' \| bb' \rightarrow B(oa'b')$
A3.7.7 $o \neq a \wedge a \neq b \wedge B(oab) \wedge o \neq o' \rightarrow B(oF(oo'a)F(oo'b))$
A3.7.8 $c \neq a \vee a = b \rightarrow ab \equiv aT(abc) \wedge (B(acT(abc)) \vee B(aT(abc)c))$
A3.7.9 $ab \equiv ab' \wedge bc \equiv b'c' \wedge ac \equiv ac' \wedge B(abc) \rightarrow B(ab'c')$
A3.7.10 $o \neq a \wedge a \neq b \wedge B(oab) \wedge o \neq o' \rightarrow B(oR'(oo'a)R'(oo'b))$
A3.7.11 $a \neq b \wedge c \neq a \wedge F(abc) = a \rightarrow B(caT(cba))$
A3.7.12 $L(a_0a_1x) \wedge L(a_0a_1y) \rightarrow O(\sigma(a_0a_1)a_0Q(x,y))$
A3.7.13 $L(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a \rightarrow O(abc) \vee O(bca) \vee O(cab)$
A3.7.14 $Hl(a'_0a_2H(a_1a'_0\sigma(a'_0a_1))$
A3.7.15 $a_1P(yxa'_0) \equiv a'_2P(yH(xy\sigma(yx)a'_o) \wedge \neg(Hl(a'_0a'_2P(yxa'_0))$
$\wedge Hl(a'_0a_1H(xy\sigma(yx))) \wedge \neg(O(a'_2a'_0P(yxa'_0)) \wedge O(a_1a'_0H(xy\sigma(yx))))$
A3.7.16 $O(abc) \rightarrow Hl(bH(ab\sigma(ba))H(abc))$
A4.5.1(k) $a_0 \neq a_1 \wedge a_0 \neq a_2 \wedge R'(a_0a_1a_2) = \sigma(a_0a_2) \wedge a_1f(a_2,y_k) \equiv a_0f(a_1,x_k)$

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