Lecture 14

Mathematical Induction
Announcements

• Mid-term: Friday, March 3 (study guide will be provided)
  – time for mid-term, 1 hour/70min
• Quiz 2: Monday, Feb 27 (study guide will be provided)
  – time, 45 min
Mathematical Induction Principle

- Given a statement $P(n)$ that depends on a natural number $n$, and whose validity we want to prove for all possible values of $n$, we proceed in the following two steps:
  - Base case: prove that $P(1)$ holds
  - Inductive step: Prove that "if $P(n)$ is true, then $P(n+1)$ is true" for all natural numbers $n$
    - $P(n)$ is called the induction hypothesis
    - These two conditions prove $P(m)$ for all $m$ (why?)

- Variation: $n$ may start from a number different from 1, e.g., 8
Mathematical Induction (a.k.a. Weak Induction)

• How do we prove that $1+2+\ldots+n = \frac{n(n+1)}{2}$?

  - Base case:
    
    \[
P(1) = \frac{1 \times 2}{2} = 1
    \]

  - Inductive step:
    
    \[
P(n) \Rightarrow P(n+1)
    \]
    \[
P(n+1) = P(n) + (n+1) = \frac{n(n+1)}{2} + (n+1)
    \]
    \[
    = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}
    \]
Example: Mathematical Induction

- Claim: Any set of n horses have the same color.
- Proof by mathematical induction?
  - Let P(n) denote “any set of n horses have the same color”
  - Take any set containing n+1 horses \{h↓1, ..., h↓n, h↓n+1\}
  - Consider the first n horses \{h↓1, ..., h↓n\}. By I.H. they all have the same color.
  - Consider the last n horses \{h↓2, ..., h↓n+1\}. Again, by I.H., they all have the same color.
  - Since h↓2, ..., h↓n are common in both sets, we conclude that h↓1, h↓2, ..., h↓n, h↓n+1 have the same color.

$$P(1)$$  
Inductive step

$$P(n) \Rightarrow P(n+1)$$

h↓1, h↓2, ..., h↓n = c↓1

h↓1, h↓2, ..., h↓n, h↓n+1 = c↓2