Propositional Logic

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Introduction to Logic

- Importance of logic to computer science
- What is logic
- Mathematical language of logic
Schedule for today

- Turning sentences into logic formulas
- Syntax of propositional logic
- Parse tree
Declarative vs. Non-Declarative

- The sum of the numbers 3 and 5 equals 8
- Could you please pass me the cup?
- A dog has four legs
- Wake up!
- Every even natural number greater than 2 is the sum of two prime numbers

Declarative sentences (sentences that you can tell whether they are true or false)

Non-Declarative sentences
## Logic Formulas

<table>
<thead>
<tr>
<th>Declarative sentences</th>
<th>Symbols</th>
<th>Connectives</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>- The sum of the numbers 3 and 5 equals 8</td>
<td>p</td>
<td>∧ (and)</td>
<td>p ∧ q → p ∨ q</td>
</tr>
<tr>
<td>- A dog has four legs</td>
<td>q</td>
<td>∨ (or)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>→ (imply)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>¬ (not)</td>
<td></td>
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</tbody>
</table>
Example (1)

If the train arrives late and there are no taxis at the station then John is late for his meeting. John is not late for his meeting. The train did arrive late.

True or False: there was a taxi at the station (and prove)

We know about:

- \( p \wedge \neg q \rightarrow r \)
- \( p \)
- \( \neg r \)

| \( p \) | The train is late |
| \( q \) | There are taxis at the station |
| \( r \) | John is late for his meeting |
If it is raining and Jane does not have her umbrella with her, then she will get wet.

Jane is not wet

It is raining

True or false: Jane has her umbrella with her (and prove)

We know about: 
- \( p \land \neg q \rightarrow r \)
- \( p \)
- \( \neg r \)
Two examples have the same structure

<table>
<thead>
<tr>
<th></th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>The train is late</td>
<td>It is raining</td>
</tr>
<tr>
<td>$q$</td>
<td>There are taxis at the station</td>
<td>Jane has her umbrella with her</td>
</tr>
<tr>
<td>$r$</td>
<td>John is late for his meeting</td>
<td>Jane gets wet</td>
</tr>
</tbody>
</table>

- If $p$ and not $q$, then $r$ \( (p \land \neg q \rightarrow r) \)
- not $r$ \( (\neg r) \)
- $p$

Therefore $q$
Syntax of Propositional Logic

Alphabet of Propositional Logic

- Propositional signature: a set of symbols, i.e., atoms.
  - E.g., p, q, r

- Propositional connectives:
  - binary: $\land$ (conjunction), $\lor$ (disjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence)
  - unary: $\neg$ (negation)
  - $\bot$ (false) and $\top$ (true)

- parentheses
A **propositional formula** of a signature is defined recursively as follows:

- every atom is a formula
- both $\bot$ (false) and $\top$ (true) are formulas
- if $F$ is a formula then $(\neg F)$ is a formula
- for any binary connective $\odot$, if $F$ and $G$ are formulas then $(F \odot G)$ is a formula

These are well-formed propositional formulas
Knowledge Checkers

Are these Well-Formed Propositional Formulas?

- $(\neg) \lor p \lor q$
- $(())$
- $(p)()$
- $\neg p$
- $(\bot)$
- $\neg p \lor q$
- $((\neg p) \lor q)$
- $(((\neg p) \lor q)))$
Knowledge Checkers

Is the last connective applied unique given a well-formed formula?
Definition of subformula

A string of consecutive symbols within a given formula to be a subformula of the given formula if it is itself a formula

$$(((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r))))$$

Is a formula of a sub-formula itself?
Subformulas are the formulas corresponding to the subtrees of the parse tree.

Priority of connectives: \( \neg, \land, \lor, \rightarrow \)

\[ \neg p \land q \rightarrow p \land (q \lor \neg r) \]
Knowledge Checkers

\[ p \rightarrow q \land p \lor q \land \neg r \]
To turn sentences into logic formulas:
- Declarative vs. non-declarative sentences
- Using connectives to connect logic symbols

Syntax of Propositional Logic
- Alphabet
  • Atoms
  • Connectives
  • Parentheses
- Recursive definition of propositional formulas

Parse tree reveals the structure; unique for well-formed formulas