Hidden Markov Model

- 2018/03/20

➢ Announcement:

- Slides for this lecture are here
- Homework 4 released, due March 26
- Quiz 7 released, due March 22
- Midterm grades ready; pick up exam papers during your TA’s office hours

Slides are largely based on information from [http://ai.berkeley.edu](http://ai.berkeley.edu) and Russel
Midterm statistics

- Mean: 21.64
- Median: 22
- Standard Deviation: 5.54
- Max: 30
- Min: 7
Last time

• Probability basics
  ➢ Random Variables
  ➢ Joint and Marginal Distributions
  ➢ Conditional Distribution
  ➢ Product Rule, Chain Rule, Bayes’ Rule
  ➢ Inference
  ➢ Independence

• Markov model

• Required reading (red means it will be on your exams):
  o R&N: Chapter 15.1-2
Markov model recap

Explicit assumption for all $t$:

$$X_t \perp \ X_1, \ldots, X_{t-2} \ | \ X_{t-1}$$

Consequently, joint distribution can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2) \cdots P(X_T|X_{T-1})$$

$$= P(X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})$$

Implied conditional independencies: (try to prove this!)

- Past variables independent of future variables given the present
  - i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp X_{t_3} \ | \ X_{t_2}$

Additional explicit assumption: $P(X_t \ | \ X_{t-1})$ is the same for all $t$

- Basic conditional independence:
  - Each time step only depends on the previous step
  - Past variables independent of future variables given the present
  - This is called the (first order) Markov property
Markov Chain Example

- States: $X = \{\text{rain, sun}\}$
  - Initial distribution: 1.0 sun
  - CPT $P(X_t | X_{t-1})$:

| $X_{t-1}$ | $X_t$ | $P(X_t | X_{t-1})$ |
|-----------|-------|-------------------|
| sun       | sun   | 0.9               |
| sun       | rain  | 0.1               |
| rain      | sun   | 0.3               |
| rain      | rain  | 0.7               |

Two new ways of representing the same CPT
Markov Chain Example

Initial distribution: 1.0 sun

What is the probability distribution after one step?

\[
P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})
\]

\[
0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9
\]
Mini-forward

- Question: What’s P(X) on some day t?

\[ P(x_1) = \text{known} \]

\[ P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t) \]

\[ = \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \]

Forward simulation
### Running Mini-forward

From initial observation of sun

\[
\begin{bmatrix}
1.0 \\ 0.0
\end{bmatrix}, \quad \begin{bmatrix}
0.9 \\ 0.1
\end{bmatrix}, \quad \begin{bmatrix}
0.84 \\ 0.16
\end{bmatrix}, \quad \begin{bmatrix}
0.804 \\ 0.196
\end{bmatrix}
\]

\[
P(X_1), \quad P(X_2), \quad P(X_3), \quad P(X_4)
\]
Ghostbuster with random movement
Ghostbuster with random movement

After 1 forward step
Ghostbuster with random movement

After several forward steps
Ghostbuster with circular movement
Ghostbuster with circular movement

After 1 forward step
Ghostbuster with circular movement

After several forward steps
Ghostbuster with circular movement

After several more forward steps
Running Mini-forward

- From initial observation of sun

\[
\begin{bmatrix}
1.0 & 0.9 & 0.84 & 0.804 \\
0.0 & 0.1 & 0.16 & 0.196 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
0.75 \\
0.25 \\
\end{bmatrix}
\]

\[P(X_1) \quad P(X_2) \quad P(X_3) \quad P(X_4) \quad P(X_\infty)\]

- From initial observation of rain

\[
\begin{bmatrix}
0.0 & 0.3 & 0.48 & 0.588 \\
1.0 & 0.7 & 0.52 & 0.412 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
0.75 \\
0.25 \\
\end{bmatrix}
\]

\[P(X_1) \quad P(X_2) \quad P(X_3) \quad P(X_4) \quad P(X_\infty)\]

- From yet another initial distribution \(P(X_1)\):

\[
\begin{bmatrix}
p \\
1-p \\
\end{bmatrix} \rightarrow \begin{bmatrix}
0.75 \\
0.25 \\
\end{bmatrix}
\]

\[P(X_1) \quad P(X_\infty)\]
Stationary distribution

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

#### Stationary distribution:

- The distribution we end up with is called the stationary distribution $P_\infty$ of the chain.

- It satisfies

$$P_\infty(X) = P_{\infty + 1}(X) = \sum_x P(X|x)P_\infty(x)$$
Stationary distribution

• Question: What’s $P(X)$ at time $t = \infty$?

$$P_\infty(\text{sun}) = P(\text{sun}|\text{sun})P_\infty(\text{sun}) + P(\text{sun}|\text{rain})P_\infty(\text{rain})$$

$$P_\infty(\text{rain}) = P(\text{rain}|\text{sun})P_\infty(\text{sun}) + P(\text{rain}|\text{rain})P_\infty(\text{rain})$$

$$P_\infty(\text{sun}) = 0.9P_\infty(\text{sun}) + 0.3P_\infty(\text{rain})$$

$$P_\infty(\text{rain}) = 0.1P_\infty(\text{sun}) + 0.7P_\infty(\text{rain})$$

$$P_\infty(\text{sun}) = 3P_\infty(\text{rain})$$

$$P_\infty(\text{rain}) = 1/3P_\infty(\text{sun})$$

Also: $P_\infty(\text{sun}) + P_\infty(\text{rain}) = 1$  

$$P_\infty(\text{sun}) = 3/4$$

$$P_\infty(\text{rain}) = 1/4$$
Application of Stationary Distribution: Web Link Analysis

- **PageRank over a web graph**
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. $c$, uniform jump to a random page (dotted lines, not all shown)
    - With prob. $1-c$, follow a random outlink (solid lines)

- **Stationary distribution**
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)
Outline for today

• Hidden Markov model
  o Components of HMM
  o Independencies in HMM
  o HMM applications
  o Inference in HMM

• Required reading (red means it will be on your exams):
  o R&N: Chapter 15.3
Pacman with sensing
Pacman with sensing
Hidden Markov Model

Markov chains not so useful for most agents
Need observations to update your beliefs

Hidden Markov models (HMMs)
Underlying Markov chain over states $X$
You observe outputs (effects) at each time step
Components of HMM -- weather

- An HMM is defined by:
  - Initial distribution: \( P(X_1) \)
  - Transitions: \( P(X_t \mid X_{t-1}) \)
  - Emissions: \( P(E_t \mid X_t) \)

\[
P(X_t \mid X_{t-1})
\]

\[
\begin{array}{c|c|c}
R_t & R_{t+1} & P(R_{t+1} \mid R_t) \\
\hline
+r & +r & 0.7 \\
+r & -r & 0.3 \\
-r & +r & 0.3 \\
-r & -r & 0.7 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
R_t & U_t & P(U_t \mid R_t) \\
\hline
+r & +u & 0.9 \\
+r & -u & 0.1 \\
-r & +u & 0.2 \\
-r & -u & 0.8 \\
\end{array}
\]
Components of HMM -- Ghostbuster

- \( P(X_1) = \text{uniform} \)

- \( P(X|X') = \) usually move clockwise, but sometimes move in a random direction or stay in place

- \( P(R_{ij}|X) = \) same sensor model as before: red means close, green means far away.
HMM for Ghostbuster
Joint distribution of HMM

- Joint distribution:
  \[ P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3) \]

- More generally:
  \[ P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t) \]

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?
Chain rule and HMM

From the chain rule, every joint distribution over $X_1, E_1, X_2, E_2, X_3, E_3$ can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2)$$

$$P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

Assuming that

$$X_2 \perp E_1 \mid X_1, \quad E_2 \perp X_1, E_1 \mid X_2, \quad X_3 \perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp X_1, E_1, X_2, E_2 \mid X_3$$

gives us the expression posited on the previous slide:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$
Chain rule and HMM

- From the chain rule, *every* joint distribution over $X_1, E_1, \ldots, X_T, E_T$ can be written as:
  \[
P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_1, E_1, \ldots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \ldots, X_{t-1}, E_{t-1}, X_t)
  \]

- **Assuming** that for all $t$:
  - State independent of all past states and all past evidence given the previous state, i.e.:
    \[
    X_t \perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}
    \]
  - Evidence is independent of all past states and all past evidence given the current state, i.e.:
    \[
    E_t \perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t
    \]
  
gives us the expression posited on the earlier slide:
  \[
P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)
  \]
Implied conditional independencies

- Many implied conditional independencies, e.g.,
  \[ E_1 \perp X_2, E_2, X_3, E_3 \mid X_1 \]
- To prove them
  - Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
  - Approach 2: directly from the graph structure
  - Intuition: If path between U and V goes through W, then \( U \perp V \mid W \)
Conditional independencies

- HMMs have two important independence properties:
  - Markov hidden process: future depends on past via the present
  - Current observation independent of all else given current state

- Quiz: does this mean that evidence variables are guaranteed to be independent?
  - [No, they tend to correlated by the hidden state]
HMM applications

• Speech recognition HMMs:
  – Observations are acoustic signals (continuous valued)
  – States are specific positions in specific words (so, tens of thousands)

• Machine translation HMMs:
  – Observations are words (tens of thousands)
  – States are translation options

• Robot tracking:
  – Observations are range readings (continuous)
  – States are positions on a map (continuous)
Filtering and Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t | e_1, ..., e_t)$ (the belief state) over time.

- We start with $B_1(X)$ in an initial setting, usually uniform.

- As time passes, or we get observations, we update $B(X)$.

- The Kalman filter was invented in the 60’s and first implemented as a method of trajectory estimation for the Apollo program.
Localization

Example from Michael Pfeiffer

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.
Localization

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake
Localization

\[ t = 2 \]
Localization

t=3
Localization

t=4

Prob

0 1
Localization

t=5
Inference

\[ P(X_1|e_1) \]

\[
P(x_1|e_1) = \frac{P(x_1, e_1)}{P(e_1)}
\]

\[
\propto_{x_1} P(x_1, e_1)
\]

\[
= P(x_1)P(e_1|x_1)
\]

\[ P(X_2) \]

\[
P(x_2) = \sum_{x_1} P(x_1, x_2)
\]

\[
= \sum_{x_1} P(x_1)P(x_2|x_1)
\]
Passage of time

- Assume we have current belief \( P(X \mid \text{evidence to date}) \)

\[
B(X_t) = P(X_t \mid e_{1:t})
\]

- Then, after one time step passes:

\[
P(X_{t+1} \mid e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t \mid e_{1:t})
= \sum_{x_t} P(X_{t+1} \mid x_t, e_{1:t}) P(x_t \mid e_{1:t})
= \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})
\]

- Basic idea: beliefs get “pushed” through the transitions
  - With the “b” notation, we have to be careful about what time step \( t \) the belief is about, and what evidence it includes

\[
B'(X_{t+1}) = \sum_{x_t} P(X' \mid x_t) B(x_t)
\]
Passage of time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})} \propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} \mid X_{t+1}) B'(X_{t+1})$$

- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize
• As we get observations, beliefs get reweighted, uncertainty “decreases”

\[ B(X) \propto P(e|X)B'(X) \]
Weather HMM

\[ R_t \quad R_{t+1} \quad P(R_{t+1} | R_t) \quad R_t \quad U_t \quad P(U_t | R_t) \]

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The forward algorithm

- We are given evidence at each time and want to know
  \[ B_t(X) = P(X_t|e_{1:t}) \]

- We can derive the following updates

\[
P(x_t|e_{1:t}) \propto \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})
= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)
= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})
\]
Online belief update

Every time step, we start with current $P(X |$ evidence)
We update for time:

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

We update for evidence:

$$P(x_t|e_{1:t}) \propto x P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

The forward algorithm does both at once (and doesn’t normalize)
Pacman with sensing
Pacman with sensing
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